A family of fourth-order q-logarithmic equations

Mario Bukal[a,*]

[a] *University of Zagreb, Faculty of Electrical Engineering and Computing, Unska 3, 10000 Zagreb, Croatia*

Abstract

We prove the existence of global in time weak nonnegative solutions to a family of nonlinear fourth-order evolution equations, parametrized by a real parameter $q \in (0, 1]$, which includes the well known thin-film ($q = 1/2$) and the Derrida–Lebowitz–Speer–Spohn (DLSS) equation ($q = 1$), subject to periodic boundary conditions in one spatial dimension. In contrast to the gradient flow approach in [25], our method relies on dissipation property of the corresponding entropy functionals (Tsallis entropies) resulting in required a priori estimates, and extends the existence result from [25] to a wider range of the family members, namely to $0 < q < 1/2$. Generalized Beckner-type functional inequalities yield an exponential decay rate of (relative) entropies, which in further implies the exponential stability in the L^1-norm of the constant steady state. Finally, we provide illustrative numerical examples supporting the analytical results.

Keywords: fourth-order diffusion equations, Tsallis entropy, entropy–entropy dissipation method, existence of weak solutions, long-time behaviour of solutions
2000 MSC: 35K30, 35B40, 35B45, 35Q99

1. Introduction

The study object of this paper is Cauchy problem to a family of nonlinear fourth-order evolution equations

$$\partial_t u = -\left(u^{2-q}\left(\log_q u\right)_{xx}\right)_{xx}, \tag{1}$$

subject to the periodic boundary conditions, i.e. $x \in \mathbb{T} \simeq [0, 1)$, and given nonnegative initial datum $u(\cdot, 0) = u_0 \in L^1(\mathbb{T})$. The family is parametrized by a real parameter q, where $\log_q$ denotes the so called q-logarithm [30] defined by

$$\log_q u = \begin{cases} \log u\,, & u > 0 \text{ and } q = 1\,, \\ \frac{u^{1-q}-1}{1-q}\,, & u > 0 \text{ and } q \neq 1\,. \end{cases} \tag{2}$$

*Corresponding author
Email address: `mario.bukal@fer.hr` (Mario Bukal)

The definition of q-logarithm as a generalization of the natural logarithm invokes mostly physical reasons. It appears in the context of the Tsallis statistics [29, 30], a non-extensive generalization of the classical Boltzmann–Gibbs statistical mechanics. Basic ingredient of the Tsallis statistics is an entropic expression, parametrized by a real parameter α,

$$\mathcal{S}_\alpha(u) = \frac{1}{\alpha - 1}\left(1 - \int u(x)^\alpha \mathrm{d}x\right)$$

called *Tsallis entropy*, which generalizes the Boltzmann entropy $\mathcal{H}(u) = -\int u \log u \,\mathrm{d}x$ recovered at the limit when $\alpha \to 1$. Tailored to macroscopic description of long-range interacting physical systems exhibiting power-law behaviour, Tsallis statistics found applications in diverse disciplines ranging from natural sciences to medicine and economics [18].

In this paper we focus on the range of parameters $q \in (0, 1]$, since it will allow us to rigorously carry out a thorough analysis — construction of global-in-time weak nonnegative solutions and long-time behaviour analysis — of equation (1). Periodic boundary conditions are taken in order to emphasize the structure of the equation without being concered about boudaries and in order to "freely" integrate by parts. For these resons we could also take homogeneous Neumann and no-flux boundary conditions, they would yield the same results. For $q = 1$, equation (1) becomes the well known Derrida–Lebowitz–Speer–Spohn (DLSS) equation

$$\partial_t u = -\left(u\left(\log u\right)_{xx}\right)_{xx}\,, \tag{3}$$

which has been first established by Derrida et al. [15] in studying of interface fluctuations in the Toom model, and later on it has been recognized as a quantum correction of the classical drift-diffusion model describing the transport of charged particles in quantum semiconductors [13]. The DLSS equation has been the subject of many papers showing its rich mathematical structure, in particular, we refer to the existence of local-in-time positive classical solutions [6] and the existence of global in time nonnegative weak solutions, as well as their long-term behaviour [24]. Setting $q = 1/2$ in (1), simple calculus (assuming enough smoothness of solutions) reveals the famous thin-film equation

$$\partial_t u = -2\left(u^{3/2}\left(u^{1/2} - 1\right)_{xx}\right)_{xx} = -\left(u u_{xxx}\right)_x \tag{4}$$

describing the evolution of the fluid thickness in the Hele-Shaw cell [11]. This equation is commonly studied within another family of fourth-order evolution equations — thin film equations

$$\partial_t u = -\left(u^\beta u_{xxx}\right)_x$$

arising in lubrication approximation of various physical models of thin viscous fluids [5, 27]. The literature on analysis of these equations is huge, thus we refer only to several prominent references [3, 4, 12]. Let us still mention that the only thin-film equation possessing the symmetry structure of the spatial differential operator as in (1) is equation (4), i.e. when $\beta = 1$. Other thin-film equations ($\beta > 0$) allow for the symmetry structure in the leading order term of the operator, but also retain a second-order perturbation term (cf. Section 6).

For smooth positive solutions, equation (1) can be rewritten in equivalent conservation law form

$$\partial_t u = -\frac{2}{3-2q}\left(u\left(u^{1/2-q}\big(u^{3/2-q}\big)_{xx}\right)_x\right)_x, \tag{5}$$

which first appeared in [14], where Denzler and McCann related these equations to a family of porous medium equations

$$\partial_t u = (u(\log_q u)_x)_x = \frac{1}{2-q}\left(u^{2-q}\right)_{xx}, \tag{6}$$

and constructed special solutions on $\mathbb{R}\times(0,\infty)$. Thorough analysis of equation (5) has been undertaken in [25] showing rigorously a gradient flow structure. Namely, for $1/2 \le q \le 1$ equation (5), set on $\mathbb{R}\times(0,\infty)$, constitutes the gradient flow of the *generalized Fisher information*

$$\mathcal{F}_q(u) = \frac{2}{(3-2q)^2}\int_{\mathbb{R}}\left(u^{3/2-q}\right)_x^2 \mathrm{d}x$$

with respect to the L^2–Wasserstein distance. Thus, the existence of weak solutions accompanied with the long time asymptotics to the stationary profiles has been established. In particular, such results were known before for the DLSS ($q = 1$) [20] and for the thin film ($q = 1/2$) equation [19]. Recently, McCann and Seis [26] took further investigations on the long time behaviour of nonnegative solutions constructed in [25]. Linearizing formally a rescaled version of equation (5) around its stationary profile (corresponding Barenblatt-type profile) and using so called entropy-information relation, which relates the generalized Fisher information $\mathcal{F}_q$ to entropy $\mathcal{H}_q$ (see eq. (7) below) and its L^2–Wasserstein gradient, they provided complete spectral information (eigenvalues with the corresponding eigenfunctions) about displacement Hessian of $\mathcal{F}_q$ in terms of the spectral information of the porous medium displacement Hessian, i.e. displacement Hessian of $\mathcal{H}_q$. Based on that powerful heuristics and known spectral information of the porous medium displacement Hessian, they conjecture a complete asymptotic expansion of solutions to equation (5) for large times.

Our approach is somewhat different, rather complementary to [25], and closely follows the methods developed in [22, 24]. It is more elementary in the sense that a complete analysis relies on a priori estimates conducted by the dissipation property of certain functionals (also called *entropies*[1] or *Lyapunov functionals*) along solutions to (1). Natural Lyapunov functional for equation (1) reads

$$\mathcal{H}_q(u) = \frac{1}{2-q}\int_{\mathbb{T}}\left(u\log_q u - \log_{q-1} u\right)\mathrm{d}x, \tag{7}$$

which for $q = 1$ reveals the classical Boltzmann–Gibbs entropy $\mathcal{H}_1(u) = \int_{\mathbb{T}}(u\log u - u + 1)\,\mathrm{d}x$. Note that if u is a probability density, then up to the factor $-1/(2-q)$, (7) equals to the *Tsallis*

[1] These entropies are strictly speaking no longer entropies in the physical sense, but nonnegative functionals with the Lyapunov property.

entropy of order $\alpha = 2 - q$. Hence, the main motivation to study again equation (5) was the discovery of its novel *entropy structure*, i.e. equation (1). Namely, equation (1) can be written as

$$\partial_t u = -\left(u^{2-q}\left(\frac{\delta\mathcal{H}_q(u)}{\delta u}\right)_{xx}\right)_{xx},$$

with $\delta\mathcal{H}_q(u)/\delta u = \log_q u$, which immediately reveals the dissipation property of functional $\mathcal{H}_q$ assuming appropriate boundary conditions. This structure has been used in [8] for the thin-film equation (4) when studying the long-time asymptotics of strong solutions, and motivated the autor to look for the same structure in other family members of (5), and furthermore, to explore the structure for the existence and long-time behaviour analysis. Key estimate for our analysis is the *entropy dissipation inequality*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}_q(u) + \kappa_q \int_{\mathbb{T}} \left(\frac{1}{\gamma^2}(u^\gamma)_{xx}^2 + \frac{16}{\gamma^4}(u^{\gamma/2})_x^4\right)\mathrm{d}x \leq 0\,, \tag{8}$$

where $\gamma = 2 - 3q/2 > 0$ and $\kappa_q > 0$ strictly positive constant given in Proposition 1 below. This estimate motivates to rewrite equation (1) in a novel form

$$\partial_t u = -\frac{1}{\gamma}\left(u^{1-q/2}(u^\gamma)_{xx}\right)_{xx} + \frac{2-q}{2\delta^2}\left((u^\delta)_x^2\right)_{xx}\,, \tag{9}$$

with $\delta = 3/2 - q > 0$, which is equivalent to (1) for smooth positive solutions, and it will provide the meaning to our notion of weak solutions.

Theorem 1. *Let $0 < q \leq 1$, $u_0 \in L^1(\mathbb{T})$ be a nonnegative function of finite entropies $\mathcal{H}_q(u_0) < \infty$ and $\mathcal{H}_1(u_0) < \infty$, and unit mass $\int_{\mathbb{T}} u_0(x)\mathrm{d}x = 1$. Let $T > 0$ be given arbitrary terminal time, then there exists a nonnegative function $u \in W^{1,m}(0,T;Y)$ satisfying $u^\gamma \in L^2(0,T;W^{2,r}(\mathbb{T}))$ and the following weak form of (9):*

$$\int_0^T \langle\partial_t u, \phi\rangle \mathrm{d}t + \int_0^T\!\!\int_{\mathbb{T}} \left(\frac{1}{\gamma}u^{1-q/2}(u^\gamma)_{xx} - \frac{2-q}{2\delta^2}(u^\delta)_x^2\right)\phi_{xx}\,\mathrm{d}x\,\mathrm{d}t = 0 \tag{10}$$

for all test functions $\phi \in L^n(0,T;W^{2,r'}(\mathbb{T}))$. Exponents m, n, r, and r' depend only on parameter q, and are explicitely writen in the proof in Section 3, while Y denotes the dual of the Sobolev space $W^{2,r'}(\mathbb{T})$.

The existence result is obtained by following a somewhat standard procedure, which proved its efficiency on several heavily nonlinear problems (cf. [7, 22, 24]). Starting from the original equation (1), which enjoys a symmetry property in the spatial differential operator, we first solve the time discrete problem by means of entropic and elliptic regularization, application of the Leray-Schauder fixed point theorem, and identification of the limit function of the *deregularization* process as a weak solution to the semi-discrete equation. Key ingredients of the procedure are entropy dissipation inequality (8) and Sobolev embeddings, while additional a priori estimates on discrete time derivatives (cf. Proposition 5 from Appendix) are required in order to perform the limit of the vanishing time-discretization. Note as well that our existence result extends the one from [25] in dimension one, in the sense that we also provide the notion of weak solutions for $0 < q < 1/2$. It is worth to emphasize at this point that despite of the lack of the comparison principle for higher-order equations in general, see for instance [1] as well as numerical examples in Section 5 which show violation of the comparison principle, equation (1) preserves the nonnegativity of global weak solutions for every $q \in (0, 1]$. Regarding the regularity issue, it has been proven in [6] for the DLSS equation ($q = 1$), using a semigroup approach, that mild solutions are arbitrary regular (smooth) as far as they remain strictly positive. For other equations ($0 < q < 1$), for which we have the existence of weak solutions, regularity is more subtle, since these equations are both: degenerate in the leading order term of the spatial differential operator and singular in lower order terms (see eq. (14) below). Additional investigations will be needed in order to gain some quantitative results concerning regularity issues.

Concerning the long time behaviour of equation (1), first observe that constant functions are stationary solutions. It has been proven in [22] that for the DLSS equation ($q = 1$), $u_\infty \equiv 1$ is exponentially stable in the sense of (relative) entropies as well as in the L^1-norm. Employing generalized Beckner-type functional inequalities, Poincaré and Csiszár-Kullback-Pinsker inequality, analogous results for $0 < q < 1$ are presented here, which are, to the best of the autor's knowledge, new.

Theorem 2. *Let $u_0 \in L^1(\mathbb{T})$ be a nonnegative unit mass function of finite entropies $\mathcal{H}_q(u_0) < \infty$ and $\mathcal{H}_1(u_0) < \infty$. For $0 < q < 1$, let u be the weak solution to (9) in the sense of Theorem 1, then*

$$\mathcal{H}_q(u(t)) \leq \mathcal{H}_q(u_0) e^{-2\lambda_q t}, \quad t \geq 0, \tag{11}$$

where $\lambda_q > 0$ is a positive constant given explicitly by (47) in Section 4. Solution u also converges exponentially in the L^1-norm to the constant steady state

$$\|u(t) - 1\|_{L^1} \leq \sqrt{2\mathcal{H}_q(u_0)} e^{-\lambda_q t}, \quad t \geq 0. \tag{12}$$

The paper is organized as follows. In Section 2 we first discuss formal dissipation properties by finding a whole family of Lyapunov functionals which are dissipated along smooth positive solutions of equation (1) and proving the key dissipation inequality (8). Section 3 and 4 are devoted to proofs of Theorem 1 and 2, respectively, while Section 5 numerically illustrates behaviour of solutions to (1) for various values of parameter q. The manuscript is concluded with several remarks regarding possible extensions of the results, and Appendix which comprises auxiliary results from the literature tailored to the situation at hand.

2. Formal dissipation properties

Assuming the existence of smooth and strictly positive solutions to equation (1), we discuss their formal dissipation structure by finding a family of functionals of the form

$$\begin{aligned} \mathcal{E}_\alpha(u) &= \frac{1}{\alpha(\alpha-1)}\int_{\mathbb{T}}(u^\alpha - \alpha u + \alpha - 1)\mathrm{d}x\,, \quad \alpha \neq 0, 1\,, \\ \mathcal{E}_1(u) &= \int_{\mathbb{T}}(u\log u - u + 1)\mathrm{d}x\,, \quad \alpha = 1\,, \\ \mathcal{E}_0(u) &= \int_{\mathbb{T}}(u - \log u)\mathrm{d}x\,, \quad \alpha = 0\,, \end{aligned} \tag{13}$$

which satisfy the Lyapunov property, i.e. $(\mathrm{d}/\mathrm{d}t)\,\mathcal{E}_\alpha(u(t)) \leq 0$ along solutions to (1) for all $t > 0$. Observe that for all $\alpha \in \mathbb{R}$ the integrand $\psi_\alpha(s)$ in (13) is nonnegative for all $s > 0$, thus, $\mathcal{E}_\alpha(u) = \int_{\mathbb{T}^d}\psi_\alpha(u)\mathrm{d}x$ is nonnegative functional for all $\alpha \in \mathbb{R}$. At this point we do not restrict the range of parameters q in (1). In order to find all $\alpha \in \mathbb{R}$ for which $\mathcal{E}_\alpha$ are Lyapunov to (1), we directly apply the method developed in [21], where an exhaustive algorithmic approach for searching for entropies is proposed by solving the corresponding polynomial decision problem.

To start with, we rewrite equation (1) in an equivalent (for smooth positive solutions) expanded form

$$\partial_t u = -\left(u^{2-2q}\left(u_{xxx} + (2-4q)\frac{u_{xx}u_x}{u} - q(1-2q)\frac{u_x^3}{u^2}\right)\right)_x\,, \tag{14}$$

and in the following lines explain very briefly the main ideas, leaving the details to the reader. Firstly, from (14) we identify the polynomial representation of the spatial differential operator

$$\partial_t u = \left(u^{3-2q}P_q\left(\frac{u_x}{u}, \frac{u_{xx}}{u}, \frac{u_{xxx}}{u}\right)\right)_x \tag{15}$$

with

$$P_q(\xi_1, \xi_2, \xi_3) = -\xi_3 - (2-4q)\xi_1\xi_2 + q(1-2q)\xi_1^3\,. \tag{16}$$

The entropy production of $\mathcal{E}_\alpha$ then reads

$$-\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_\alpha(u(t)) = \int_{\mathbb{T}} u^{\alpha+2-2q}\left(\frac{u_x}{u}\right)P_q\left(\frac{u_x}{u}, \frac{u_{xx}}{u}, \frac{u_{xxx}}{u}\right)\mathrm{d}x\,, \tag{17}$$

and the integrand is then formally represented by another polynomial $S_q(\xi_1, \xi_2, \xi_3, \xi_4) = \xi_1 P_q(\xi_1, \xi_2, \xi_3)$. In order to achieve the integral inequality $-(\mathrm{d}/\mathrm{d}t)\mathcal{E}_\alpha(u(t)) \geq 0$, we systematically use integration by parts and operate on integrands using their polynomial representation. Following [21], basic integration by parts formulae applied to (17) are represented by so called *shift polynomials:*

$$\begin{aligned} T_1(\xi) &= (\alpha - 1 - 2q)\xi_1^4 + 3\xi_1^2\xi_2\,, \\ T_2(\xi) &= (\alpha - 2q)\xi_1^2\xi_2 + \xi_1\xi_3 + \xi_2^2\,, \\ T_3(\xi) &= (\alpha + 1 - 2q)\xi_1\xi_3 + \xi_4\,, \end{aligned}$$

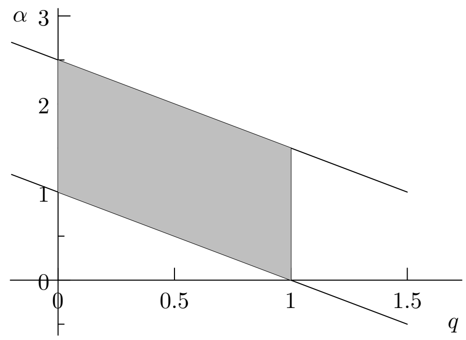


Figure 1: Range of entropies (parameters α) versus equation parameter q.

while all other can be obtained as linear combinations of these. Exhaustive application of integration by parts formulae to prove the sign of the entropy dissipation then amounts to a decision problem

$$\exists\,(c_1, c_2, c_3)\,,\ \forall\,\xi \in \mathbb{R}^4\,,\ (S_q + c_1T_1 + c_2T_2 + c_3T_3)(\xi) \geq 0$$

about the nonnegativity of the polynomial $S_q + c_1T_1 + c_2T_2 + c_3T_3$. This polynomial represents all possible integrands equivalent to S_q in the sense of equality of the integral in (17). Indefinite terms which appear in the above poynomial and cannot be controlled by any other terms are $c_3\xi_4$ and $(c_2 + c_3(\alpha + 1 - 2q) - 1)\xi_1\xi_3$. That ultimatively requires $c_3 = 0$ and $c_2 = 1$, thus leading to a simplified decision problem

$$\exists\,c_1\,,\ \forall\,\xi \in \mathbb{R}^2\,,\ \left(q(1-2q) + c_1(\alpha - 1 - 2q)\right)\xi_1^4 + (3c_1 + \alpha - 2 + 2q)\xi_1^2\xi_2 + \xi_2^2 \geq 0\,.$$

The latter can be solved directly using some computer algebra system like `Mathematica` or even manually applying [21, Lemma 11], and results in the following algebraic relations between parameters α and q (cf. Figure 1.):

$$1 - q \leq \alpha \leq \frac{5}{2} - q\,. \tag{18}$$

By the above procedure we have proven:

Theorem 3. *Let $u : \mathbb{T} \times (0,\infty) \to \mathbb{R}$ be a smooth and strictly positive solution to equation (1), then functionals $\mathcal{E}_\alpha$ defined by (13) are Lyapunov functionals for the equation for all $\alpha \in \mathbb{R}$ satisfying (18).*

Remark. Result of the previous theorem can even be strengthened in the sense that if $\alpha < 1 - q$ or $\alpha > 5/2 - q$, then $\mathcal{E}_\alpha$ is *not* a Lyapunov functional. The proof follows directly employing [21, Theorem 19].

The following corollary, which will be used in the construction of weak solutions, is a direct consequence of the Lyapunov property of the natural entropy $\mathcal{E}_1$ for equation (1) when $0 \leq q \leq 1$.

Corollary 1. *Let $0 \le q \le 1$ and $u \in H^2(\mathbb{T})$ strictly positive then*

$$\int_{\mathbb{T}} u^{2-q} (\log_q u)_{xx} (\log u)_{xx} \mathrm{d}x \ge 0\,. \tag{19}$$

Among all Lyapunov functionals $\mathcal{E}_\alpha$ to (1), we particularly utilize the dissipation property of $\mathcal{E}_{2-q}$, previously (and later on) denoted by $\mathcal{H}_q$. If u is a smooth strictly positive solution to equation (1), then integration by parts implies the dissipation of $\mathcal{H}_q$ according to

$$-\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}_q(u(t)) = \int_{\mathbb{T}} u^{2-q} \left(\log_q u\right)^2_{xx} \mathrm{d}x\,. \tag{20}$$

Using the same techniques as of the above algorithmic search for entropies, we even provide lower bounds on the entropy production (20), which will play a crucial role in the construction of weak solutions to equation (1), i.e. (9), and in the study of their long time behaviour.

Proposition 1 (Entropy dissipation inequalities)**.** *Let $u \in H^2(\mathbb{T})$ be strictly positive, then the following entropy dissipation inequalities hold:*

$$\int_{\mathbb{T}} u^{2-q} \left(\log_q u\right)^2_{xx} \mathrm{d}x \ge \kappa_q \int_{\mathbb{T}} \left(\frac{1}{\gamma^2}(u^\gamma)^2_{xx} + \frac{16}{\gamma^4}(u^{\gamma/2})^4_x\right) \mathrm{d}x\,, \tag{21}$$

$$\int_{\mathbb{T}} u^{2-q} \left(\log_q u\right)^2_{xx} \mathrm{d}x \ge \frac{\tilde{\kappa}_q}{\gamma^2} \int_{\mathbb{T}} (u^\gamma)^2_{xx} \mathrm{d}x\,, \tag{22}$$

where $\kappa_q = 4/(52 - 24q + 9q^2)$, and $\tilde{\kappa}_q = 4/(16 - 24q + 9q^2)$ for $q \in (0, 2/3]$ and $\tilde{\kappa}_q = 1$ for $q \in (2/3, 1]$.

Proof. The left hand side of integral inequality (21) is represented by $R_q(\xi) = \xi_2^2 - 2q\xi_1\xi_2 + q^2\xi_1^4$, while the right hand side is represented by $Q_q(\xi) = \xi_2^2 + (2-3q)\xi_1^2\xi_2 + (2-3q+9q^2/4)\xi_1^4$. Employing systematic integration by parts formulae leads to the decision problem

$$\exists\,(c_1, c_2, c_3)\,,\ \forall\,\xi \in \mathbb{R}^4\,,\ (R_q - \kappa Q_q + c_1 T_1 + c_2 T_2 + c_3 T_3)(\xi) \ge 0$$

with $\alpha = 2 - q$ and questing after optimal $\kappa > 0$. Resolving the latter polynomial amounts to the simplified decision problem

$$\exists\,c_1\,,\ \forall\,\xi \in \mathbb{R}^4\,,\ \left(q^2 + c_1(1-3q) - \kappa\Big(2 - 3q + \frac{9}{4}q^2\Big)\right)\xi_1^4 \\ + (3c_1 - 2q - \kappa(2-3q))\xi_1^2\xi_2 + (1-\kappa)\xi_2^2 \ge 0\,,$$

whose solution is algebraic relation

$$\kappa \le \frac{4}{52 - 24q + 9q^2}\,,$$

which immediately provides optimal κ. The proof of the second inequality follows the same, but solving a slightly modified decision problem. □

Remark. Let $u : \mathbb{T} \times (0,\infty) \to \mathbb{R}$ be a smooth and strictly positive solution to equation (1) with $q \in (0,1]$, then we have formally proven the following entropy production inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}_q(u(\cdot,t)) + \frac{\tilde{\kappa}_q}{\gamma^2}\int_{\mathbb{T}} (u^\gamma)_{xx}^2 \, \mathrm{d}x \le 0\,, \quad t > 0\,.$$

Using the Beckner-type inequality, entropy production bound can be related to the entropy itself (cf. Lemma 1 and the proof of Theorem 2) in Section 4), yielding to an evolution inequality for $\mathcal{H}_q$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}_q(u(\cdot,t)) + 2\lambda_q \mathcal{H}_q(u(\cdot,t)) \le 0\,, \quad t > 0\,,$$

which implies the exponential decay of functional $\mathcal{H}_q$ along smooth positive solutions to equation (1) at a constant rate $2\lambda_q > 0$. Using then generalized Csiszár-Kullback-Pinsker inequality (cf. Theorem 5 in Appendix), implies in further the exponential convergence of smooth positive solutions to the constant steady state in the L^1-norm at the constant rate $\lambda_q > 0$. This remark is only to briefly point out that smooth positive solutions (if such exist) exponentialy converge to the constant steady state having the same mass, i.e. constants are formally exponentially stable states of equation (1) assuming periodic boundary conditions. For the reader's convenience, the same result will be rigorously and more explanatory proved in Section 4 for weak solutions.

3. Construction of global weak solutions

This section is devised for the proof of Theorem 1 resolving step by step the construction of global-in-time weak nonnegative solutions. Since the existence proof for $q = 1$ can be found in [22, 24], here we tailor the proof for $0 < q < 1$.

3.1. Semi-discrete problem

Let a time-discretization step $\tau > 0$ be given, we start with solving the semi-discrete problem.

Proposition 2. *Let $u_0 \in L^1(\mathbb{T})$ be a nonnegative function of finite entropies $\mathcal{H}_q(u_0) < \infty$ and $\mathcal{H}_1(u_0) < \infty$ and of unit mass. Then there exists u, with $u^\gamma \in W^{2,r}(\mathbb{T})$ (recall $\gamma = 2 - 3q/2$), a weak solution to the following elliptic problem:*

$$\frac{1}{\tau}\int_{\mathbb{T}} (u - u_0)\phi \, \mathrm{d}x = -\int_{\mathbb{T}} \left(\frac{1}{\gamma} u^{1-q/2}(u^\gamma)_{xx} - \frac{2-q}{2\delta^2}(u^\delta)_x^2 \right) \phi_{xx} \, \mathrm{d}x \tag{23}$$

for all test functions $\phi \in W^{2,r'}(\mathbb{T})$, where $r' \ge 1$ denotes the Hölder conjugate of r determined below. The solution is of the unit mass and the following discrete entropy estimate holds

$$\mathcal{H}_q(u) + \kappa_q \tau \int_{\mathbb{T}} \left(\frac{1}{\gamma^2}(u^\gamma)_{xx}^2 + \frac{16}{\gamma^4}(u^{\gamma/2})_x^4 \right) \mathrm{d}x \le \mathcal{H}_q(u_0)\,, \tag{24}$$

with $\kappa_q > 0$ given in Proposition 1.

Proof. We first consider elliptic problem — semidiscretization of equation (1):

$$\frac{1}{\tau}(u-u_0) = -\left(u^{2-q}\left(\log_q u\right)_{xx}\right)_{xx} \quad \text{on } \mathbb{T}\,, \tag{25}$$

which will only be our starting point to the construction of a weak solution. To prevail the nonlinearity in (25) we introduce a new *entropy variable* $y_\varepsilon = Dh_{q,\varepsilon}(u) = \log_q u + \varepsilon \log u$ deduced from the regularized entropy density $h_{q,\varepsilon} = h_q + \varepsilon h_1$ with $\varepsilon > 0$ fixed. The new variable both linearizes equation (25) and asserts invertibility $u_\varepsilon = Dh_{q,\varepsilon}^{-1}(y_\varepsilon)$, since $D^2 h_{q,\varepsilon}(w) > 0$ for all $w > 0$. Additionally, we regularize equation (25) by subtracting an elliptic operator $\varepsilon(y_{\varepsilon,xxxx} + (u_\varepsilon^{2-q}(\log u_\varepsilon)_{xx})_{xx} + y_\varepsilon)$ on the right hand side of (25). The resulting equation then reads

$$\frac{1}{\tau}(u_\varepsilon - u_0) = -\left(\left(u_\varepsilon^{2-q} + \varepsilon\right) y_{\varepsilon,xx}\right)_{xx} - \varepsilon y_\varepsilon \quad \text{on } \mathbb{T}\,, \tag{26}$$

and will be solved in next lines by means of the Leray–Schauder fixed point theorem.

For fixed $y_\varepsilon \in L^\infty(\mathbb{T})$ let $u_\varepsilon = Dh_{q,\varepsilon}^{-1}(y_\varepsilon)$ and $\sigma \in [0,1]$, we introduce bilinear form a and linear functional f on $H^2(\mathbb{T})$ as follows:

$$\begin{aligned} a(z,\phi) &= \int_{\mathbb{T}} (\sigma u_\varepsilon^{2-q} + \varepsilon) z_{xx}\phi_{xx}\,\mathrm{d}x + \varepsilon \int_{\mathbb{T}} z\phi\,\mathrm{d}x\,, \\ f(\phi) &= -\frac{\sigma}{\tau}\int_{\mathbb{T}} (u_\varepsilon - u_0)\phi\,\mathrm{d}x\,, \qquad \forall \phi \in H^2(\mathbb{T})\,. \end{aligned}$$

Since $y_\varepsilon \in L^\infty(\mathbb{T})$ and the change of variables $Dh_{q,\varepsilon}^{-1}$ is continuous, then $u_\varepsilon^{2-q} \in L^\infty(\mathbb{T})$, and it is easily to check that a is bounded, as well as coercive

$$a(z,z) \geq \varepsilon \int_{\mathbb{T}} (z_{xx}^2 + z^2)\mathrm{d}x \geq C\varepsilon \|z\|_{H^2}^2\,,$$

due to the Poincaré inequality. Similarly, the continuity of f is also easily verified. Hence, the Lax-Milgram lemma provides the existence of a unique solution $z \in H^2(\mathbb{T})$ to the elliptic problem

$$a(z,\phi) = f(\phi)\,, \quad \phi \in H^2(\mathbb{T})\,. \tag{27}$$

Next we define the mapping $S_\varepsilon : L^\infty(\mathbb{T}) \times [0,1] \to L^\infty(\mathbb{T})$ by $S_\varepsilon(y_\varepsilon, \sigma) := z$. Continuity of S_ε follows from the Lax-Milgram lemma — z depends continuously on (u_ε, σ), and from the continuity of the change of variables — u_ε depends continuously on y_ε in the sense of the strong convergence in $L^\infty(\mathbb{T})$. The relative compactness is a direct consequence of the compact embedding $H^2(\mathbb{T}) \hookrightarrow L^\infty(\mathbb{T})$. In order to apply the Leray-Schauder fixed point theorem (version from [28]) on operator S_ε, it remains to find a closed, convex subset $B_\varepsilon \subset L^\infty(\mathbb{T})$ containing the zero element of $L^\infty(\mathbb{T})$ such that:

(ls1) $S_\varepsilon(y_\varepsilon, \sigma) \neq y_\varepsilon$ for all $y_\varepsilon \in \partial B_\varepsilon$ and $\sigma \in [0,1]$,
(ls2) $S_\varepsilon(\partial B_\varepsilon \times \{0\}) \subset B_\varepsilon$.

We shall choose

$$B_\varepsilon = \{y_\varepsilon \in L^\infty(\mathbb{T}) : \ \|y_\varepsilon\|_{L^\infty} \le \Delta(\varepsilon)\}$$

with a suitable $\Delta(\varepsilon) > 0$ determined bellow, and will proove that all solutions $y_\varepsilon \in L^\infty(\mathbb{T})$ of $S_\varepsilon(y_\varepsilon, \sigma) = y_\varepsilon$ for some $\sigma \in [0,1]$ lie in the interior of B_ε. It is easily seen that $S_\varepsilon(y_\varepsilon, 0) = 0$ for all $y_\varepsilon \in L^\infty(\mathbb{T}^d)$, hence (ls2) is easily satisfied. What remains is to prove the σ-uniform boundedness of all possible fixed points of $S_\varepsilon(\cdot, \sigma)$ for all $\sigma \in [0,1]$. Let $\sigma \in (0,1)$ be arbitrary and $\bar{y}_\varepsilon \in L^\infty(\mathbb{T})$ such that $S_\varepsilon(\bar{y}_\varepsilon, \sigma) = \bar{y}_\varepsilon$, then $\bar{y}_\varepsilon$ solves

$$\int_{\mathbb{T}} (\sigma \bar{u}_\varepsilon^{2-q} + \varepsilon)\bar{y}_{\varepsilon,xx}\phi_{xx}\mathrm{d}x + \varepsilon \int_{\mathbb{T}} \bar{y}_\varepsilon \phi \, \mathrm{d}x = -\frac{\sigma}{\tau}\int_{\mathbb{T}} (\bar{u}_\varepsilon - u_0)\phi \, \mathrm{d}x \,, \quad \phi \in H^2(\mathbb{T}) \,, \tag{28}$$

where $\bar{u}_\varepsilon = Dh_{q,\varepsilon}^{-1}(\bar{y}_\varepsilon)$. Since $\bar{y}_\varepsilon \in L^\infty(\mathbb{T})$, then $\bar{u}_\varepsilon$ is strictly positive. By construction, $S_\varepsilon(y_\varepsilon, \sigma) \in H^2(\mathbb{T})$, thus we can take the test function $\phi = \bar{y}_\varepsilon$ in (28). Then convexity of the entropy density $h_{q,\varepsilon}$ and the entropy dissipation inequalities (19) and (21) imply

$$\begin{aligned}
\frac{1}{\tau}\left(\mathcal{H}_{q,\varepsilon}(\bar{u}_\varepsilon) - \mathcal{H}_{q,\varepsilon}(u_0)\right) &\le \frac{1}{\tau}\int_{\mathbb{T}} (\bar{u}_\varepsilon - u_0)\bar{y}_\varepsilon \, \mathrm{d}x \\
&= -\int_{\mathbb{T}} \bar{u}_\varepsilon^{2-q}\big((\log_q \bar{u}_\varepsilon)_{xx}^2 + 2\varepsilon(\log_q \bar{u}_\varepsilon)_{xx}(\log \bar{u}_\varepsilon)_{xx} \\
&\qquad + \varepsilon^2(\log \bar{u}_\varepsilon)_{xx}^2\big)\mathrm{d}x - \frac{\varepsilon}{\sigma}\int_{\mathbb{T}} (\bar{y}_{\varepsilon,xx}^2 + \bar{y}_\varepsilon^2)\mathrm{d}x \\
&\le -\kappa_q \int_{\mathbb{T}} (\bar{u}_\varepsilon^\gamma)_{xx}^2 \mathrm{d}x - \frac{\varepsilon}{\sigma}\int_{\mathbb{T}} (\bar{y}_{\varepsilon,xx}^2 + \bar{y}_\varepsilon^2)\mathrm{d}x \,,
\end{aligned}$$

which yields

$$\mathcal{H}_{q,\varepsilon}(\bar{u}_\varepsilon) + \tau\kappa_q \int_{\mathbb{T}} \left((\bar{u}_\varepsilon^\gamma)_{xx}^2 + \frac{16}{\gamma^2}(\bar{u}_\varepsilon^{\gamma/2})_x^4 \right) \mathrm{d}x + \frac{\tau\varepsilon}{\sigma}\int_{\mathbb{T}} (\bar{y}_{\varepsilon,xx}^2 + \bar{y}_\varepsilon^2)\mathrm{d}x \le \mathcal{H}_{q,\varepsilon}(u_0) \,. \tag{29}$$

The last inequality provides the uniform (σ-independent) bound on $\bar{y}_\varepsilon \in H^2(\mathbb{T})$,

$$\|\bar{y}_\varepsilon\|_{H^2}^2 \le \frac{C\mathcal{H}_{q,\varepsilon}(u_0)}{\tau\varepsilon} \,.$$

and the Sobolev embedding $H^2(\mathbb{T}) \hookrightarrow L^\infty(\mathbb{T})$ asserts $\|\bar{y}_\varepsilon\|_{L^\infty} \le C/\sqrt{\tau\varepsilon}$. The latter yields the existence of $\Delta(\varepsilon) > 0$ and $B_\varepsilon \subset L^\infty(\mathbb{T})$ such that (ls2) holds. Finally, the Leray-Schauder theorem can be applied providing the existence of a fixed point y_ε to $S_\varepsilon(\cdot, 1)$, i.e. the existence of a weak solution to (26).

The entropy estimate (29) implies that $(u_\varepsilon^\gamma)_{xx}$ is ε-uniformly bounded in $L^2(\mathbb{T})$ and u_ε^{2-q} is ε-uniformly bounded in $L^1(\mathbb{T})$. The latter implies the ε-uniform bound of u_ε^γ in $L^r(\mathbb{T})$ with $r = 1 + q/(4-3q) \in (1,2)$, which together with the first assertion yields the ε-uniform bound of u_ε^γ in

$W^{2,r}(\mathbb{T})$. Therefore, up to extraction of a subsequence as $\varepsilon \downarrow 0$:

$$\begin{aligned} u_\varepsilon^\gamma &\rightharpoonup u^\gamma \quad \text{in } W^{2,r}(\mathbb{T})\,, \\ u_\varepsilon^\gamma &\to u^\gamma \quad \text{in } W^{1,\infty}(\mathbb{T})\,, \\ u_\varepsilon &\to u \quad \text{in } L^\infty(\mathbb{T})\,. \end{aligned}$$

Inequality (29) also implies the ε-uniform bound on $u_\varepsilon^{\gamma/2}$ in $W^{1,4}(\mathbb{T})$, and thus applying Proposition 4 from Appendix (cf. [23, Appendix]), we conclude the strong convergence

$$u_\varepsilon^\delta \to u^\delta \quad \text{in } W^{1,p}(\mathbb{T})\,,\ \text{with } p = 3 - 1/(3-2q) \geq 2\,.$$

The last two terms on the left hand side of (29) give that $\sqrt{\varepsilon} y_\varepsilon$ is bounded in $H^2(\mathbb{T})$, which yields

$$\varepsilon y_\varepsilon \to 0 \quad \text{in } H^2(\mathbb{T})\,.$$

Since u_ε is strictly positive and H^2-regular, we can write

$$\begin{aligned} u_\varepsilon^{2-q} y_{\varepsilon,xx} &= u_\varepsilon^{2-q} (\log_q u_\varepsilon)_{xx} + \varepsilon u_\varepsilon^{2-q} (\log u_\varepsilon)_{xx} \\ &= \frac{1}{\gamma} u_\varepsilon^{1-q/2} (u_\varepsilon^\gamma)_{xx} - \frac{2-q}{2\delta^2} (u_\varepsilon^\delta)_x^2 + \frac{\varepsilon}{\gamma} u_\varepsilon^{q/2} \left((u_\varepsilon^\gamma)_{xx} - 4 (u_\varepsilon^{\gamma/2})_x^2 \right)\,. \end{aligned} \tag{30}$$

The last identity and the above convergence results are sufficient for passing to the limit when $\varepsilon \downarrow 0$ in (26). The ε-term in (30) converges to 0 strongly in $L^r(\mathbb{T})$, and the rest converges to

$$\frac{1}{\gamma} u^{1-q/2} (u^\gamma)_{xx} - \frac{2-q}{2\delta^2} (u^\delta)_x^2 \qquad \text{weakly in } \ L^r(\mathbb{T})\,.$$

This allows us to identify the limit function u as a weak solution to (23). The mass conservation property, $\int_{\mathbb{T}} u \,\mathrm{d}x = 1$, follows directly by taking $\phi \equiv 1$ as the test function in (23). Entropy estimate (29) combined with the weak lower semicontinuity of the entropy and entropy dissipation bound reveal the discrete entropy production inequality (24)

$$\begin{aligned} &\mathcal{H}_q(u) + \tau \kappa_q \int_{\mathbb{T}} \left((u^\gamma)_{xx}^2 + \frac{16}{\gamma^2} (u^{\gamma/2})_x^4 \right) \mathrm{d}x \\ &\quad \leq \liminf_{\varepsilon \downarrow 0} \left(\mathcal{H}_{q,\varepsilon}(u_\varepsilon) + \tau \kappa_q \int_{\mathbb{T}} \left((u^\gamma)_{xx}^2 + \frac{16}{\gamma^2} (u^{\gamma/2})_x^4 \right) \mathrm{d}x \right) \\ &\quad \leq \liminf_{\varepsilon \downarrow 0} \left(\mathcal{H}_{q,\varepsilon}(u_0) \right) = \mathcal{H}_q(u_0)\,, \end{aligned}$$

which finishes the proof of Proposition 2. □

3.2. Passage to the limit $\tau \downarrow 0$

Let terminal time $T > 0$ be such that $N = T/\tau \in \mathbb{N}$. Using the previous semi-discrete procedure we recursively construct solutions (u_τ^k) solving elliptic problems from Proposition 2

$$\frac{1}{\tau}(u - u_\tau^{k-1}) = -\frac{1}{\gamma}\big(u^{1-q/2}(u^\gamma)_{xx}\big)_{xx} + \frac{2-q}{2\delta^2}\Big((u^\delta)_x^2\Big)_{xx} \quad \text{on } \mathbb{T}\,, \tag{31}$$

for $k = 1, \dots, N$, and define piecewise constant function $u_\tau : (0,T) \to L^1(\mathbb{T})$ by

$$u_\tau(t) := u_\tau^k \quad \text{for } (k-1)\tau < t \le k\tau,\ k = 1, \dots, N; \qquad u_\tau(0) = u_0\,. \tag{32}$$

Multiplying (31) by a test function $\phi(t) \in W^{2,r'}(\mathbb{T})$, for $(k-1)\tau < t \le k\tau$, and integrating over $(0,T)$ (summing up) yields

$$\frac{1}{\tau}\int_0^T\!\!\int_{\mathbb{T}}(u_\tau - \sigma_\tau u_\tau)\phi\,\mathrm{d}x\mathrm{d}t + \int_0^T\!\!\int_{\mathbb{T}}\left(\frac{1}{\gamma}u_\tau^{1-q/2}(u_\tau^\gamma)_{xx} - \frac{2-q}{2\delta^2}(u_\tau^\delta)_x^2\right)\phi_{xx}\,\mathrm{d}x\,\mathrm{d}t = 0 \tag{33}$$

for all test functions $\phi \in L^1(0,T;W^{2,r'}(\mathbb{T}))$. In order to perform the limit $\tau \downarrow 0$, we need the following a priori estimate, which will provide us with the sought compactness.

Proposition 3. *There exists $C > 0$, independent of $\tau > 0$, such that*

$$\tau^{-1}\|u_\tau - \sigma_\tau u_\tau\|_{L^m(\tau,T;Y)} + \|u_\tau^\gamma\|_{L^2(0,T;W^{2,r})} \le C\,, \tag{34}$$

where $\sigma_\tau u_\tau = u_\tau(\cdot - \tau)$ denotes the τ-shift operator, $m = 2\gamma/(3-2q)$, and $Y = (W^{2,r'}(\mathbb{T}))'$ denotes the dual of the Sobolev space $W^{2,r'}(\mathbb{T})$.

Proof. Integrating (24) over the time interval $[0,T]$ we get

$$\mathcal{H}_q(u_\tau(T)) + \kappa_q\int_0^T\!\!\int_{\mathbb{T}}\left((u_\tau^\gamma)_{xx}^2 + \frac{16}{\gamma^2}(u_\tau^{\gamma/2})_x^4\right)\mathrm{d}x\mathrm{d}t \le \mathcal{H}_q(u_0)\,, \tag{35}$$

from which we conclude the following τ-uniform bounds:

$$\|(u_\tau^\gamma)_{xx}\|_{L^2(0,T;L^r)} \le C\,, \tag{36}$$

$$\|(u_\tau^{\gamma/2})_x\|_{L^4(0,T;L^4)} \le C\,, \tag{37}$$

$$\|u_\tau^\gamma\|_{L^\infty(0,T;L^r)} \le C\,. \tag{38}$$

The first and the third bound imply the desired bound $\|u_\tau^\gamma\|_{L^2(0,T;W^{2,r})} \le C$. Continuous embedding $W^{2,r}(\mathbb{T}) \hookrightarrow L^\infty(\mathbb{T})$ provides the uniform bound $\|u_\tau^\gamma\|_{L^2(0,T;L^\infty)} \le C$, which in further yields $\|u_\tau^{1-q/2}\|_{L^s(0,T;L^\infty)} \le C$ with $s = 4 - 2q/(2-q) \ge 2$.

Next we prove the uniform bound on the first term in (34), i.e. there exists a constant $C > 0$ such that

$$\frac{1}{\tau}\left|\int_0^T\!\!\int_{\mathbb{T}}(u_\tau - \sigma_\tau u_\tau)\varphi(x,t)\mathrm{d}x\mathrm{d}t\right| \le C\|\varphi\|_{L^n(0,T;W^{2,r'})} \tag{39}$$

holds for all test functions $\varphi \in L^n(0,T;W^{2,r'}(\mathbb{T}))$, and n determined below. In order to prove (39) we discuss the boundedness of those two terms on the right hand side of (31) separately. First, using Hölder's inequality and above uniform bounds, we have

$$\begin{aligned}\left|\int_0^T\int_{\mathbb{T}} u_\tau^{1-q/2}(u_\tau^\gamma)_{xx}\varphi_{xx}\,\mathrm{d}x\mathrm{d}t\right| &\le \int_0^T \|u_\tau^{1-q/2}\|_{L^\infty}\|(u_\tau^\gamma)_{xx}\|_{L^r}\|\varphi_{xx}\|_{L^{r'}}\mathrm{d}t \\ &\le \|u_\tau^{1-q/2}\|_{L^s(0,T;L^\infty)}\|(u_\tau^\gamma)_{xx}\|_{L^2(0,T;L^r)}\|\varphi_{xx}\|_{L^{s'}(0,T;L^{r'})} \\ &\le C\|\varphi\|_{L^{s'}(0,T;W^{2,r'})}\,,\end{aligned}$$

where $n = (4-3q)/(1-q)$ is obtained from $1/n = 1/2 - 1/s$. Identity $(u_\tau^\delta)_x^2 = (4-3q)/(6-4q)u_\tau^{1-q/2}(u_\tau^{\gamma/2})_x^2$ a.e. in $\mathbb{T}\times(0,T)$ and the above uniform bounds give us

$$\|(u_\tau^\delta)_x^2\|_{L^m(0,T;L^2)} \le C\,,$$

with $m = 2\gamma/(3-2q)$ obtained from $1/m = 1/2 + 1/s$. It is easily to check that n is the Hölder conjugate of m, hence the Hölder inequality reveals

$$\begin{aligned}\left|\int_0^T\int_{\mathbb{T}}(u_\tau^\delta)_x^2\varphi_{xx}\,\mathrm{d}x\mathrm{d}t\right| &\le \|(u_\tau^\delta)_x\|^2_{L^m(0,T;L^2)}\|\varphi_{xx}\|_{L^n(0,T;L^2)} \\ &\le C\|\varphi\|_{L^n(0,T;W^{2,r'})}\,.\end{aligned}$$

□

A priori estimate (34) implies the existence (up to subsequences) of weak limits $v \in L^2(0,T;W^{2,r}(\mathbb{T}))$ and $\partial_t u \in L^m(0,T;Y)$, such that as $\tau \downarrow 0$:

$$u_\tau^\gamma \rightharpoonup v \quad \text{weakly in} \quad L^2(0,T;W^{2,r}(\mathbb{T}))\,, \tag{40}$$

$$\tau^{-1}(u_\tau - \sigma_\tau u_\tau) \rightharpoonup \partial_t u \quad \text{weakly in} \quad L^m(0,T;Y)\,. \tag{41}$$

Since $r \ge 1$, $W^{2,r}(\mathbb{T})$ embedds compactly into $W^{1,s}(\mathbb{T})$ for any $1 \le s < \infty$, Proposition 5, given in Appendix, implies the compactness of (u_τ) in $L^{2\gamma}(0,T;W^{1,s}(\mathbb{T}))$ for any $1 \le s < \infty$, i.e.

$$u_\tau \to u \quad \text{strongly in} \quad L^{2\gamma}(0,T;W^{1,s}(\mathbb{T}))\,. \tag{42}$$

The latter asserts pointwise convergences $u_\tau \to u$ a.e. and $u_\tau^\gamma \to u^\gamma$ a.e., which finally allow to identify $v = u^\gamma$. Again, applying Proposition 4, we obtain

$$u_\tau^\delta \to u^\delta \quad \text{strongly in} \quad L^p(0,T;W^{1,p}(\mathbb{T}))\,. \tag{43}$$

The above convergence results are sufficient for passing to the limit in (33) as $\tau \downarrow 0$ and to identify u as a weak solution of equation (9) in the sense of Theorem 1, i.e.

$$\int_0^T \langle \partial_t u, \phi\rangle_{Y,W^{2,r'}}\mathrm{d}t + \int_0^T\int_{\mathbb{T}}\left(\frac{1}{\gamma}u^{1-q/2}(u^\gamma)_{xx} - \frac{2-q}{2\delta^2}(u^\delta)_x^2\right)\phi_{xx}\,\mathrm{d}x\,\mathrm{d}t = 0$$

for all test functions $\phi \in L^n(0,T;W^{2,r'}(\mathbb{T}))$.

4. Long time behaviour of weak solutions

As already mentioned in the introduction, the idea for this paper arose from studying the work of Carrillo and Toscani [8] on the long time asymptotics of strong solutions to the thin-film equation (4). Posing the equation on the real line and starting with a nonnegative initial datum (compactly supported or of finite second moment), they proved an algebraic decay rate in the L^1-norm of the strong solution towards the unique self-similar profile. Moreover, the same strategy could be formally applied to a broader class of fourth-order diffusion equations (cf. [8, Eq. (6.10)]), which includes our equation (1). However, we will not follow that direction, but to conclude the analysis, we turn our view to the global spatially periodic weak solutions constructed in the previous section and prove Theorem 2.

For the DLSS equation ($q = 1$), long-time behaviour of weak solutions showing an exponential decay in the L^1-norm to the constant steady state has been established in [22]. Thus we focus on parameters $q \in (0, 1)$. In lieu of the logarithmic Sobolev inequality, used in [22] for the DLSS equation, here we invoke a generalized Beckner-type inequality, recently proved in [9], which provides required entropy – entropy dissipation inequality in our case.

Lemma 1 ([9, Lemma 4])**.** *Let $1 \le r < 2$ and $p \ge 1/r$, and let $w \in H^1(\mathbb{T})$, then the following generalized Beckner-type inequality holds*

$$\|w\|_{L^r}^{2-r}\left(\int_{\mathbb{T}} |w|^r \mathrm{d}x - \left(\int_{\mathbb{T}} |w|^{1/p} \mathrm{d}x\right)^{pr}\right) \le C_B \|\partial_x w\|_{L^2}^2 \tag{44}$$

where $C_B = r(pr-1)/(4\pi^2(2-r))$.

Proof of Theorem 2. For $\tau > 0$, let $u_\tau^1, u_\tau^2, \dots$ be recursively constructed sequence of unit mass solutions as in (31). The discrete entropy estimate (24)[2] gives

$$\mathcal{H}_q(u_\tau^k) + \tilde{\kappa}_q \tau \int_{\mathbb{T}} \left(\frac{1}{\gamma^2}(u_\tau^{k\,\gamma})_{xx}^2\right) \mathrm{d}x \le \mathcal{H}_q(u_\tau^{k-1})\,, \quad k \ge 1 \tag{45}$$

with $\tilde{\kappa}_q$ given in (22). Employing the Beckner-type inequality (44) for $w = (u_\tau^k)^\gamma$, $r = (2-q)/\gamma$ and $p = \gamma$ we infer

$$\left(\int_{\mathbb{T}} (u_\tau^k)^{2-q} \mathrm{d}x\right)^{(2-r)/r} \mathcal{H}_q(u_\tau^k) \le \frac{C_B}{(2-q)(1-q)} \int_{\mathbb{T}} (u_\tau^{k\,\gamma})_x^2 \,\mathrm{d}x\,. \tag{46}$$

Furthermore, invoking the Poincaré inequality in (46) yields the discrete entropy – entropy dissipation inequality

$$\left(\int_{\mathbb{T}} (u_\tau^k)^{2-q} \mathrm{d}x\right)^{(2-r)/r} \mathcal{H}_q(u_\tau^k) \le \frac{C_B}{4\pi^2(2-q)(1-q)} \int_{\mathbb{T}} (u_\tau^{k\,\gamma})_{xx}^2 \,\mathrm{d}x\,,$$

[2] In fact we can take an improved estimate analogous to (22).

where $(4\pi^2)^{-1}$ arises as the square of the optimal Poincaré constant on $\mathbb{T}$ [16]. Combining the latter with (45) gives a recursive inequality

$$\mathcal{H}_q(u_\tau^k) + 2\lambda_q^k \tau \mathcal{H}_q(u_\tau^k) \leq \mathcal{H}_q(u_\tau^{k-1}), \quad k \geq 1,$$

where

$$\lambda_q^k = \frac{2\pi^2 \tilde{\kappa}_q (2-q)(1-q)}{\gamma^2 C_B} \|u_\tau^k\|_{L^{2-q}}^{(2-r)\gamma}.$$

Since $1 = \|u_0\|_{L^1} = \|u_\tau^k\|_{L^1} \leq \|u_\tau^k\|_{L^{2-q}}$, we conclude that $\lambda_q^k \geq \lambda_q$, where

$$\lambda_q = \frac{2\pi^2 \tilde{\kappa}_q (2-q)(1-q)}{\gamma^2 C_B} = \frac{16\pi^4 \tilde{\kappa}_q (1-q)}{\gamma^2}, \tag{47}$$

and thus

$$\mathcal{H}_q(u_\tau^k) + 2\lambda_q \tau \mathcal{H}_q(u_\tau^k) \leq \mathcal{H}_q(u_\tau^{k-1}), \quad k \geq 1.$$

The latter inductively implies

$$\mathcal{H}_q(u_\tau(t)) \leq (1 + 2\lambda_q \tau)^{-t/\tau} \mathcal{H}_q(u_0)$$

for all $t \in ((k-1)\tau, k\tau]$, and u_τ defined as in (32). Since $u_\tau(t) \to u(t)$ pointwise a.e. as $\tau \downarrow 0$ and $(1 + 2\lambda_q \tau)^{-t/\tau}$ converges to $\exp(-2\lambda_q t)$, thus on the limit $\tau \downarrow 0$ we find

$$\mathcal{H}_q(u(t)) \leq \mathcal{H}_q(u_0) e^{-2\lambda_q t}, \quad t > 0.$$

Applying the Csiszár-Kullback-Pinsker inequality [2] (see Theorem 5 in Appendix)

$$\|u(t) - 1\|_{L^1} \leq \sqrt{2\mathcal{H}_q(u(t))}, \quad t > 0,$$

finishes the proof. □

Remark. For unit mass solutions, i.e. $\int_{\mathbb{T}} u \, \mathrm{d}x = 1$, entropy $\mathcal{H}_q$ coincides with the *relative entropy* defined by

$$\mathcal{H}_{q,rel}(u|u_\infty) = \frac{1}{2-q} \int_{\mathbb{T}} u \log_q \Big(\frac{u}{u_\infty}\Big) u_\infty^{1-q} \, \mathrm{d}x. \tag{48}$$

Omitting relative entropies before, we simplified the technical aspect of the previous discussion, but the decay results from Theorem 2 straightforwardly extend to non-unit mass solutions involving relative entropies instead. In that case, however, convergence rate $\tilde{\lambda}_q$ depends on the mass of the initial data (cf. expression for λ_q^k above)

$$\tilde{\lambda}_q = \frac{16\pi^4 \tilde{\kappa}_q (1-q)}{\gamma^2} \|u_0\|_{L^1}^{2(1-q)}. \tag{49}$$

Numerical example of the next section illustrates this analytical observation.

5. Illustrative numerics

For illustration purposes, equation (1) is solved using finite differences. Let $x_i = ih$, $i = 0,\dots,N-1$ be an equidistant grid on the one-dimensional torus $\mathbb{T} \simeq [0,1)$, let $t_k = k\tau$, $k \in \mathbb{N}$ and let U_i^k be approximation of $u(x_i, t_k)$. Setting periodic boundary conditions $U_l^k = U_{l \bmod N}^k$ for all $l \in \mathbb{Z}$ and $k \in \mathbb{N}$, implicit Euler in time and central difference discretization in space of equation (1) yield the following nonlinear system of algebraic equations

$$\frac{1}{\tau}(U_i^k - U_i^{k-1}) = -\delta^{\langle 2\rangle}\left((U_i^k)^{2-q}\delta^{\langle 2\rangle}\log_q U_i^k\right), \quad i = 0,\dots,N,\ k \geq 1, \tag{50}$$

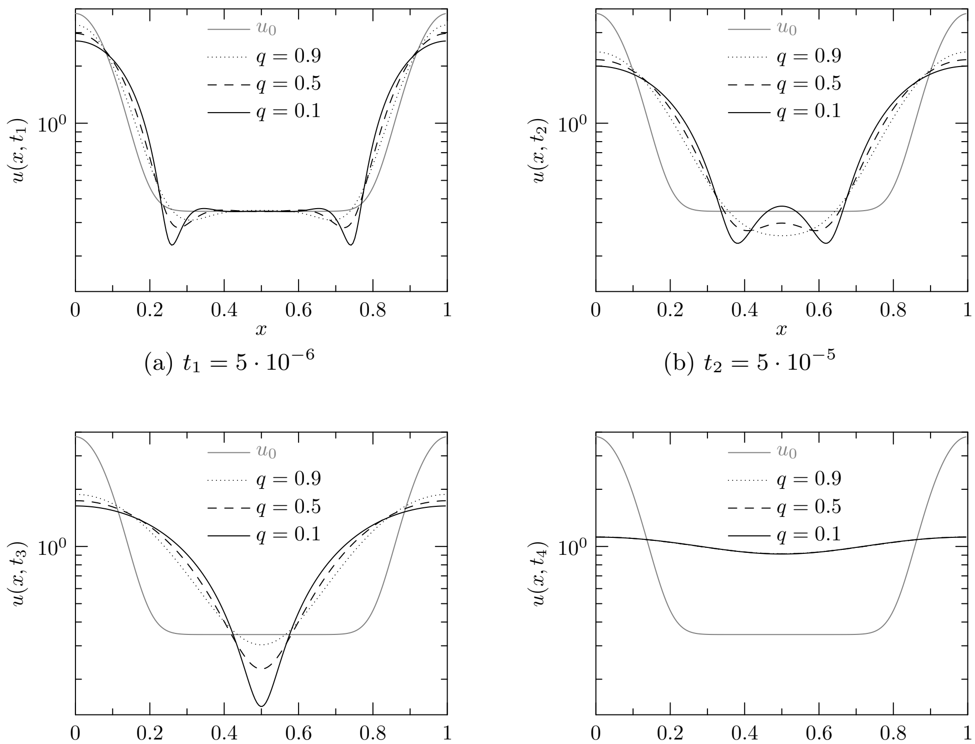
where $\delta^{\langle 2\rangle}U_l^k = (U_{l+1}^k - 2U_l^k + U_{l-1}^k)/h^2$ denotes the second-order central difference operator. The nonlinear system in unknown $U^k = (U_0^k,\dots,U_{N-1}^k) \in \mathbb{R}^N$ is then solved using Newton's method with typically 3 to 4 iterations and as initial guess we take U^{k-1}, the solution from the previous time step. In all our computations we take the time discretization step $\tau = 10^{-7}$ and the grid resolution $h = 0.005$.

Example 1 (unit mass solutions). For the first illustrative example we take initial datum $u_0(x) = (\cos(\pi x)^{16} + 0.1)/M$, where $M = 0.2909$ is the normalizing constant such that u_0 is of the unit mass. Then we solve system (50) for three different values of parameter q: 0.1, 0.5, and 0.9. Figure 2 shows the evolution of the numerical solution to (1) starting with the initial datum u_0. Numerical solutions have been computed at four time instants: $t_1 = 5\cdot 10^{-6}$ (Figure 2a), $t_2 = 5\cdot 10^{-5}$ (Figure 2b), $t_3 = 2\cdot 10^{-4}$ (Figure 2c), and $t_4 = 1.5\cdot 10^{-3}$ (Figure 2d). One observes differences in numerical evolution for different parameters q, but eventually they all converge to the same constant steady state $u_\infty = 1$. Figure 3 reveals numerically the exponential decay of the corresponding entropies $\mathcal{H}_q$.

Example 2 (non-unit mass solutions). By this example we consider numerical solutions to equation (1) starting with initial datum $u_0(x) = \cos(\pi x)^{16} + 0.1$, which is of non-unit mass $M = 0.2909$. Figures 4a–4d below show numerical solutions for the same parameters $q = 0.1, 0.5, 0.9$ and at the same time instants $t_1,\dots,t_4$ as above. Furthermore, Figure 5 reveals an exponential decay of the corresponding relative entropies $\mathcal{H}_{q,rel}$, defined by (48), but also significant differences in convergence rates, which reflect the modified convergence rate (49), i.e. its dependence on the mass and parameter q.

6. Concluding remarks

In this paper we analyzed a family of nonlinear fourth-order evolution equations (1), which can be considered as a fourth-order generalization of the porous medium equations. Contrary to the previously developed gradient flow approach [25], which recognizes these equations as gradient flows of generalized Fisher information with respect to the L^2-Wasserstein distance, our approach relates these equations to the Tsallis entropies of order $2-q$. Based on the dissipation property of the Tsallis entropy (upon convenient change of the sign), we proved the existence of global in time weak nonnegative spatially periodic solutions for the range of parameters $q \in (0,1]$, extending the



(a) $t_1 = 5 \cdot 10^{-6}$

(b) $t_2 = 5 \cdot 10^{-5}$

(c) $t_3 = 2 \cdot 10^{-4}$

(d) $t_4 = 1.5 \cdot 10^{-3}$

Figure 2: Numerical evolution of equation (1) for unit mass initial datum u_0.

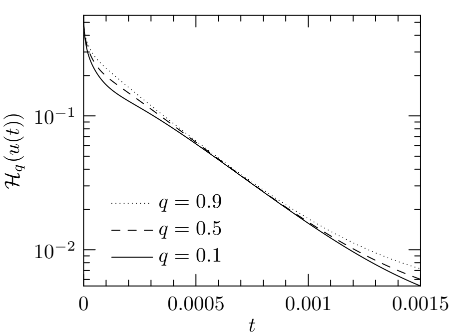
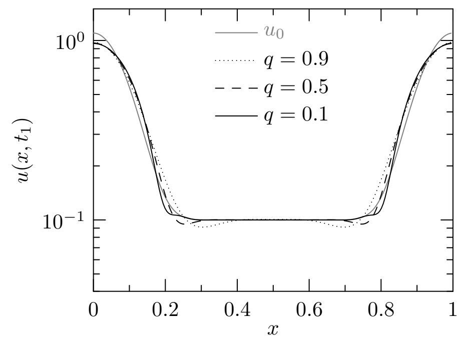
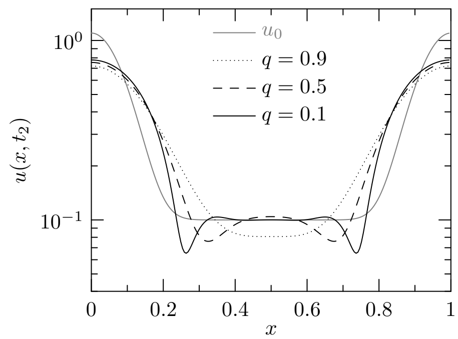


Figure 3: Entropy decay (logarithmic scale).

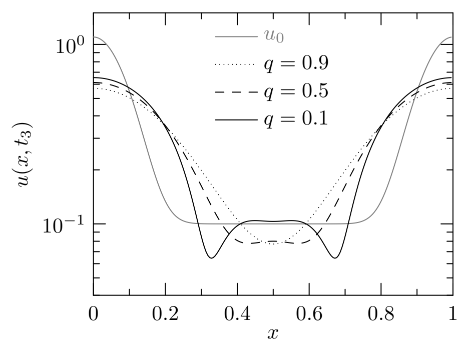
existence result from [25] to the wider range of parameters, namely to $0 < q < 1/2$. Additionally, using generalized Beckner-type functional inequalities, we also provide an exponential decay of (relative) entropies, as well as the exponential stability in the L^1-norm of the constant steady state.



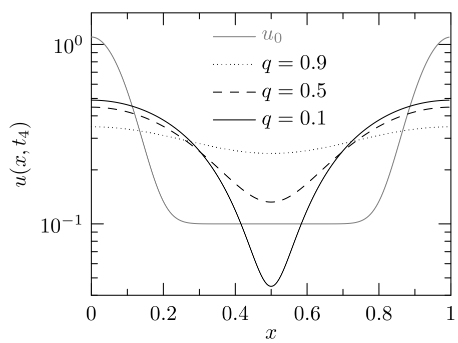
(a) $t_1 = 5 \cdot 10^{-6}$



(b) $t_2 = 5 \cdot 10^{-5}$



(c) $t_3 = 2 \cdot 10^{-4}$



(d) $t_4 = 1.5 \cdot 10^{-3}$

Figure 4: Numerical evolution of equation (1) for non-unit mass initial datum u_0.

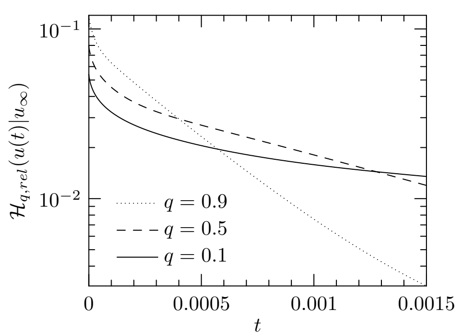


Figure 5: Relative entropy decay (logarithmic scale).

To conclude the paper, in the following we briefly comment on several questions which have not been touched here, but deserve to be discussed in more detail.

Multi-dimensional case. In the paper we only discussed the one-dimensional equation. Spatially

d-dimensional equation (1), i.e. $\log_q$-form of the fourth-order equation from [25] reads (cf. [22])

$$\partial_t u = -\sum_{i,j=1}^{d} \partial_{ij}^2 \left(u^{2-q} \partial_{ij}^2 \left(\log_q u \right) \right) + \frac{1}{2} \sum_{i,j=1}^{d} \left(\partial_{ij}^2 (u^{2-2q}) \partial_{ij}^2 u - \partial_{ii}^2 (u^{2-2q}) \partial_{jj}^2 u \right) .$$

We expect that the above one-dimensional analysis extends to the multi-dimensional case, at least in physically relevant dimensions $d = 2, 3$, where Sobolev embeddings are still strong enough.

(Non)-Uniqueness. It is known for the DLSS equation ($q = 1$) that there exists a countable family of time-independent weak solutions

$$\bar{u}(x,t) = \cos^2(n\pi x) , \quad x \in \mathbb{T} , \, t > 0 , \; n \in \mathbb{N} ,$$

which obviously do not converge to the constant steady state [22]. However, the weak solutions constructed in [22, Theorem 1], which dissipate the physical entropy and converge to the constant steady state, gain certain regularity and according to [17] these are unique in that class of regularity. Whether similar story applies to equations (1) for $0 < q < 1$, is still an open question.

Derivation. Equations (1) appeared in the literature [14] in the conservation law form (5), and in the sense of the gradient flow world they can be interpreted as fourth-order porous medium equations. However, these equations have not been yet derived from physical grounds. In analogy with the DLSS equation, it is the author's belief that equations (1) could also be interpreted as "higher-order approximations" of some non-local porous medium–type models.

More general fourth-order diffusion equations. To which extent is our method applicable to more general fourth-order diffusion equations proposed in [8]

$$\partial_t u = -\left(\Phi(u) \left(h'(u) \right)_{xx} \right)_{xx} ,$$

where Φ is increasing from $\Phi(0) = 0$, and is related to the nonnegative convex function h by $\Phi'(u) = uh''(u)$? This question might be answered with additional assumptions on Φ and h.

Relation to other thin-film equations. General fourth-order thin-film equation with parameter $\beta > 0$ can be written in a form with symmetrized leading order term of the spatial differential operator, but a nonlinear second-order term retains, unless $\beta = 1$,

$$\partial_t u = -\left(u^\beta u_{xxx} \right)_x = -\left(u^{3\beta/2} \left(\log_{\beta/2} u \right)_{xx} \right)_{xx} - \frac{\beta(\beta-1)}{2} \left(u^{\beta-2} u_x^3 \right)_x .$$

Whether this insight brings any novelty into the studying of thin-film equations is not clear at the moment and could be the subject of some future investigations.

Appendix

First, we recall a particular variant of the Leray–Schauder theorem that has been proven in [28].

Theorem 4 (Leray–Schauder)**.** *Let X be a Banach space and let $B \subset X$ be a closed and convex set such that the zero element of X is contained in the interior of B. Furthermore, let $S : B \times [0,1] \to X$ be a continuous map such that its range $S(B \times [0,1])$ is relatively compact in X. Assume that $S(x,\sigma) \neq x$ for all $x \in \partial B$ and $\sigma \in [0,1]$ and that $S(\partial B \times \{0\}) \subset B$. Then there exists $x_0 \in B$ such that $S(x_0, 1) = x_0$.*

Next result is proved in [23, Appendix], and we use it to obtain strong convergence of the sequence (u_ε^δ) in Section 3.1, provided strong convergence of the sequence (u_ε^γ) and a uniform bound on $(u_\varepsilon^{\gamma/2})$, in notation of Section 3.

Proposition 4. *Let $0 < \beta < \gamma < \alpha < \infty$, $1 < p, q, r < \infty$ be given, where $\alpha p = \beta q = \gamma r$. Assume that (u_n) is a sequence of strictly positive functions on $\mathbb{T}$ with the following properties:*

1. *u_n^α converges strongly to u^α in $W^{1,p}(\mathbb{T})$, and*
2. *u_n^β is bounded in $W^{1,q}(\mathbb{T})$.*

Then u_n^γ converges strongly to u^γ in $W^{1,r}(\mathbb{T})$. The respective result holds for sequences of nonnegative functions $u_n : (0,T) \times \mathbb{T} \to \mathbb{R}$ upon replacing $W^{1,s}(\mathbb{T})$ by $L^s(0,T; W^{1,s}(\mathbb{T}))$ for, respectively, $s = p, q, r$.

A variant of the Aubin-Lions-Dubinskiĭ lemma, which provides compactness of a sequence of piecewise constant functions in certain parabolic spaces, is our next auxiliary tool. The following proposition is a direct analogue of Theorem 3 from [10], but which involves higher-order Sobolev spaces instead. The proof follows exactly the same lines as in [10], hence we omit it here.

Proposition 5. *Let (u_τ) be a sequence of nonnegative functions which are constant on each subinterval $((k-1)\tau, k\tau]$, $1 \leq k \leq N$, $T = N\tau$, and let $0 < \gamma < \infty$ and $1 < r < \infty$. Let $Y = (W^{2,r'}(\mathbb{T}))'$ denotes the dual of the Sobolev space $W^{2,r'}(\mathbb{T})$, where r' is the Hölder conjugate of r. If there exists a positive constant $C > 0$, independent of $\tau > 0$, such that*

$$\tau^{-1} \| u_\tau - \sigma_\tau u_\tau \|_{L^1(\tau,T;Y)} + \| u_\tau^\gamma \|_{L^2(0,T;W^{2,r})} \leq C \,, \tag{51}$$

then (u_τ) is relatively compact in $L^{2\gamma}(0,T; W^{1,s}(\mathbb{T}))$ for any $1 \leq s \leq \infty$.

Finally, we give a generalized version of the Csiszár-Kullback-Pinsker inequality whose proof can be found in [2, Section 2.2].

Theorem 5. *Let $\Omega \subset \mathbb{R}^d$ be a domain and $u, v \in L^1(\Omega)$ satisfy $u \geq 0$, $v > 0$, and $\int_\Omega u(x)\mathrm{d}x = \int_\Omega v(x)\mathrm{d}x = 1$. Let $\psi \in C^0([0,+\infty) \cap C^4(0,+\infty)$ be such that $\psi(1) = 0$, $\psi''(1) + \psi'''(1) > 0$, ψ is convex, and $1/\psi''$ is concave on $(0,+\infty)$. Then*

$$\| u - v \|_{L^1(\Omega)}^2 \leq \frac{2}{\psi''(1)} \int_\Omega \psi\Big(\frac{u}{v}\Big) v \,\mathrm{d}x.$$

It is straightforward to check that functions $\psi_q(s) = (s^{2-q} - (2-q)s + 1 - q)/((2-q)(1-q))$ satisfy the above conditions exactly for $q \in (0,1)$. Taking $\Omega = \mathbb{T}$ and $v = 1$, immediately reveals

$$\|u - 1\|^2_{L^1(\mathbb{T})} \leq 2\mathcal{H}_q(u) \quad \text{for all } q \in (0,1)\,.$$

Acknowledgment. This work has been supported by the Croatian Science Foundation under Grant agreement No. 9477 (MAMPITCoStruFl). The author is very thankful to Prof. A. Jüngel (TU Wien) and Prof. D. Matthes (TU München) for their valuable remarks and encouraging dicusussions.

References

[1] P. Álvarez-Caudevilla and V. A. Galaktionov. Well-posedness of the Cauchy problem for a fourth-order thin-film equation via regularization approaches. *Nonlinear Analysis* 121 (2015), 19–35.

[2] A. Arnold, P. Markowich, G. Toscani, and A. Unterreiter. On convex Sobolev inequalities and the rate of convergence to equilibrium for Fokker-Planck type equations. *Comm. Part. Diff. Eq.* 26 (2001), 43–100.

[3] J. Becker and G. Grün. The thin-film equation: Recent advances and some new perspectives. *J. Phys.: Condens. Matter* 17 (2005), 291–307.

[4] F. Bernis and A. Friedman. Higher order nonlinear degenerate parabolic equations. *J. Diff. Eqs.* 83 (1990), 179–206.

[5] A. Bertozzi. The mathematics of moving contact lines in thin liquid films. *Notices Amer. Math. Soc.* 45 (1998), 689–697.

[6] P. Bleher, J. Lebowitz, and E. Speer. Existence and positivity of solutions of a fourth-order nonlinear PDE describing interface fluctuations. *Commun. Pure Appl. Math.* 47 (1994), 923–942.

[7] M. Bukal, A. Jüngel, and D. Matthes. A multidimensional nonlinear sixth-order quantum diffusion equation. *Ann. Inst. H. Poincaré Anal. non linéaire* 30 (2013), 337–365.

[8] J. A. Carrillo, G. Toscani. Long-Time Asymptotics for Strong Solutions of the Thin Film Equation. *Commun. Math. Phys.* 225 (2002), 551–571.

[9] C. Chainais-Hillairet, A. Jüngel, and S. Schuchnigg. Entropy-dissipative discretization of nonlinear diffusion equations and discrete Beckner inequalities. *Math. Model. Numer. Anal.* 50 (2016), 135–162.

[10] X. Chen, A. Jüngel, and J.-G. Liu. A note on Aubin-Lions-Dubinskii lemmas. *Acta Appl. Math.* 133 (2014), 33–43.

[11] P. Constantin, T. Dupont, R. E. Goldstein, L. P. Kadanoff, M. J. Shelley, and S. M. Zhou. Droplet breakup in a model of the Hele-Shaw cell. *Phys. Rev. E* 47 (1993), 4169–4181.

[12] R. Dal Passo, H. Garcke, and G. Grün. On a fourth order degenerate parabolic equation: global entropy estimates and qualitative behaviour of solutions. *SIAM J. Math. Anal.* 29 (1998), 321–342.

[13] P. Degond, F. Méhats, and C. Ringhofer. Quantum energy-transport and drift-diffusion models. *J. Stat. Phys.* 118 (2005), 625–665.

[14] J. Denzler and R. J. McCann. Nonlinear diffusion from a delocalized source: affine self-similarity, time reversal, & nonradial focusing geometries. *Ann. Inst. H. Poincaré Anal. non linéaire* 25 (2008), 865–888.

[15] B. Derrida, J. Lebowitz, E. Speer, and H. Spohn. Fluctuations of a stationary nonequilibrium interface. *Phys. Rev. Lett.* 67 (1991), 165–168.

[16] J. Dolbeault, I. Gentil and A. Jüngel. A nonlinear fourth-order parabolic equation and related logarithmic Sobolev inequalities. *Commun. Math. Sci.* 4 (2006), 275–290.

[17] J. Fisher. Uniqueness of Solutions of the Derrida-Lebowitz- Speer-Spohn Equation and Quantum Drift-Diffusion Models. *Comm. Part. Diff. Eq.*, 38 (2013), 2004–2047.

[18] M. Gell-Mann and C. Tsallis, eds., *Nonextensive Entropy - Interdisciplinary Applications.* Oxford University Press, New York, 2004.

[19] L. Giacomelli and F. Otto. Variational formulation for the lubrication approximation of the Hele-Shaw flow. *Calc. Var. PDEs*, 13 (2001), 377–403.

[20] U. Gianazza, G. Savaré, and G. Toscani. The Wasserstein gradient flow of the Fisher information and the quantum drift-diffusion equation. *Arch. Ration. Mech. Anal.* 194 (2009), 133–220.

[21] A. Jüngel and D. Matthes. An algorithmic construction of entropies in higher-order nonlinear PDEs. *Nonlinearity* 19 (2006), 633–659.

[22] A. Jüngel and D. Matthes. The Derrida-Lebowitz-Speer-Spohn equation: existence, non-uniqueness, and decay rates of the solutions. *SIAM J. Math. Anal.* 39 (2008), 1996–2015.

[23] A. Jüngel and J.-P. Milišić. A sixth-order nonlinear parabolic equation for quantum systems. *SIAM J. Math. Anal.* 41 (2009), 1472–1490.

[24] A. Jüngel and R. Pinnau. Global non-negative solutions of a nonlinear fourth-oder parabolic equation for quantum systems. *SIAM J. Math. Anal.* 32 (2000), 760–777.

[25] D. Matthes, R. McCann, and G. Savaré. A family of nonlinear fourth order equations of gradient flow type. *Comm. Part. Diff. Eq.* 34 (2009), 1352–1397.

[26] R. McCann and C. Seis. The Spectrum of a Family of Fourth-Order Nonlinear Diffusions Near the Global Attractor. *Comm. Part. Diff. Eq.* 40 (2015), 191–218.

[27] T. Myers. Thin films with high surface tension. *SIAM Rev.* 40 (1998), 441-462.

[28] A. Potter. An elementary version of the LeraySchauder theorem. *J. Lond. Math. Soc.* 5 (1972), 414–416.

[29] C. Tsallis. Possible Generalization of Boltzmann–Gibs Statistics. *Journal of Statistical Physics* 52 (1988), 479–487.

[30] S. Umarov, C. Tsallis and S. Steinberg. On a q-Central Limit Theorem Consistent with Nonextensive Statistical Mechanics. *Milan j. math.* 76 (2008), 307–328.