Derivation of homogenized Euler–Lagrange equations for von Kármán rods

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Abstract

In this paper we study the effects of simultaneous homogenization and dimension reduction in the context of convergence of stationary points for thin nonhomogeneous rods under the assumption of the von Kármán scaling. Assuming stationarity conditions for a sequence of deformations close to a rigid body motion, we prove that the corresponding sequences of scaled displacements and twist functions converge to a limit point, which is the stationary point of the homogenized von Kármán rod model. The analogous result holds true for the von Kármán plate model.

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1. Introduction

Boosted by the rigidity result of Friesecke, James and Müller [14], the rigorous derivation of various approximate models from three-dimensional nonlinear elasticity theory and its variational justification have become a prominent research topic in the last decade. In particular, based on a refined rigidity result [15], a whole hierarchy of limiting lower-domensional models has been derived by means of Γ-convergence techniques [4, 10]. In this paper, we only refer to the derivation of nonlinear inextensible rod models [23, 25]. In all these models however, the material is assumed to be fixed, i.e. does not have a microstructure. There is also a vast literature on studying the effects of simultaneous homogenization and dimension reduction in various contexts [5, 9, 19], but we will focus on the derivation of rod models. In [21] the authors studied a linearized rod model assuming its homogeneity along the central line and nonhomogeneous microstructure in the cross section. A systematic approach combining rigidity estimates [15] and the two-scale convergence method [1] was presented in [28] for the model of bending rod under the assumption of periodic microstructure. The same homogenized model has been obtained in [22] without periodicity assumptions, while using a Γ-convergence method tailored to dimension reduction in higher-order elasticity models. This

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method has been previously applied for the derivation of homogenized von Kármán plate [32] and linearized elasticity models [8], and in this paper we briefly outline how it accomplishes the homogenized von Kármán rod model (see Section 2.5).

The main purpose of this paper is to study convergence of stationary points of thin three-dimensional inhomogeneous rods in the von Kármán scaling regime. The above mentioned Γ-convergence techniques roughly assert that a compact sequence of minimizers of scaled energies converges (on a subsequence) to a minimizer of the limit energy. However, due to nonlinearities, these minimizers are typically only global and do not necessary satisfy the corresponding Euler–Lagrange equation. Secondly, it is possible that there exist stationary points that are not minimizers and thus their convergence can not be analyzed by the Γ-convergence approach. Convergence of stationary points of thin elastic rods in the bending regime has been first studied in [26] on a simplified model of thin 2D strips and thenafter extended to the full 3D problem in [24]. In order to identify the limit equations, besides the rigidity estimate, the authors also used compensated compactness and careful truncation arguments. Later on, convergence of stationary points of thin elastic rods in higher-order scaling regimes (including the von Kármán scaling), under physical growth conditions for the elastic energy density, has been established in [11]. However, in all these models the rod material was assumed to be fixed, i.e. without a microstructure.

In this paper we allow for possibility of materials with a microstructure (including random) and study the effects of simultaneous homogenization and dimension reduction in the context of convergence of stationary points in the von Kármán rod model. Let us denote by $\Omega = (0, L) \times \omega \subset \mathbb{R}^3$ a three-dimensional rod-like canonical domain of length $L > 0$ and cross-section $\omega \subset \mathbb{R}^2$ bounded, having a Lipschitz boundary. The (scaled) energy functional of a rod of thickness $h > 0$ occupying material domain $\Omega_h = (0, L) \times h\omega$ associated to a deformation $y^h : \Omega \to \mathbb{R}^3$ is defined on the canonical domain by

$$\mathcal{E}^h(y^h) = \int_\Omega W^h(x, \nabla_h y^h)\mathrm{d}x - \int_\Omega f^h \cdot y^h \mathrm{d}x \,. \tag{1}$$

Above W^h is an elastic energy density describing an admissible composite material (see Section 2.2), $\nabla_h y^h = (\partial_1 y^h \,|\, \frac{1}{h}\partial_2 y^h \,|\, \frac{1}{h}\partial_3 y^h)$ denotes the scaled gradient of the deformation, and f^h describes an external load. It is well known that different scaling regimes with respect to the thickness h in the applied load and elastic energy lead in the limit to different rod models [15, 31]. In the *von Kármán scaling* of the rod, which is the subject of the research here, we assume that the elastic energy of a sequence (y^h) satisfies

$$\limsup_{h \downarrow 0} \frac{1}{h^4} \int_\Omega W^h(x, \nabla_h y^h)\mathrm{d}x < \infty \,. \tag{2}$$

The forcing term scales as $f^h = h^3 f$, where $f = f_2 e_2 + f_3 e_3$ with $f_2, f_3 \in L^2(0, L)$, meaning that only normal loads to the mid-fiber of the rod are considered. One can prove that under this scaling of the forces the global minimizers satisfy the assumption (2), see [15] for details.

Under assumption (2) on a sequence of deformations (y^h) one can also prove, based on the theorem of geometric rigidity [14], that there exist sequences of rotations $(\bar{R}^h) \subset \mathrm{SO}(3)$ and constants $(c^h) \subset \mathbb{R}^3$, such that transformed deformations $\hat{y}^h = (\bar{R}^h)^T y^h - c^h$ converge to the identity deformation on $(0, L)$ in the L^2-norm, i.e. $\hat{y}^h \to x_1 e_1$, and moreover, $\nabla_h \hat{y}^h \to I$ in the L^2-norm [23] (cf. Theorem 2.1 below), where I is the 3×3 identity matrix. Furthermore, the scaled displacements, defined by

$$u^h(x_1) = \int_\omega \frac{\hat{y}_1^h - x_1}{h^2}\mathrm{d}x', \qquad v_i^h(x_1) = \int_\omega \frac{\hat{y}_i^h}{h}\mathrm{d}x' \quad \text{for } i = 2, 3, \tag{3}$$

and the twist functions

$$w^h(x_1) = \frac{1}{\mu(\omega)} \int_\omega \frac{x_2\hat{y}_3^h - x_3\hat{y}_2^h}{h^2}\mathrm{d}x', \tag{4}$$

where $\mu(\omega) = \int_\omega (x_2^2 + x_3^2)\mathrm{d}x'$, converge (weakly) on a suitably extracted subsequence to $(u, v_2, v_3, w) \in H^1(0, L) \times H^2(0, L) \times H^2(0, L) \times H^1(0, L)$ (see Theorem 2.4 for more details). If one assumes the natural fixed boundary condition at one end of the rod, then it can be shown that $\bar{R}^h$ can be taken to be identity and c^h can be taken to be zero (see Remark 2.2 below).

The strain sequence (G^h) is implicitly defined through the decomposition of the scaled gradient as $\nabla_h\hat{y}^h = R^h(I + h^2G^h)$, where (R^h) denotes the sequence of rotation functions constructed in Theorem 2.1. Convergence results from Theorems 2.1 and 2.4 allow for the representation of the symmetrized strain $\operatorname{sym} G^h$ as the sum of a fixed and a corrector (and remainder) term as follows:

$$\operatorname{sym} G^h = \underbrace{\operatorname{sym}(\imath(m_d))}_{\text{fixed term}} + \underbrace{\operatorname{sym}\nabla_h\psi^h}_{\text{corrector term}} + o^h, \tag{5}$$

where the fixed term is

$$m_d = \begin{pmatrix} u' + \frac{1}{2}\left((v_2')^2 + (v_3')^2\right) - v_2''x_2 - v_3''x_3 \\ -w'x_3 \\ w'x_2 \end{pmatrix}, \tag{6}$$

with $\imath$ denoting the inclusion of vectors into 3×3 matrices, the sequence (ψ^h), called the corrector sequence, satisfies $(\psi_1^h, h\psi_2^h, h\psi_3^h) \to 0$, $\int_\omega (x_2\psi_3^h - x_3\psi_2^h)\mathrm{d}x' \to 0$ in the L^2-norm and $\|\operatorname{sym}\nabla_h\psi^h\|_{L^2(\Omega)} \le C$, while the rest sequence (o^h) converges to zero in the L^2-norm. The corrector term plays the role of the corrector in homogenization. Utilizing the Γ-convergence method developed for the bending rod model in [22], we can analogously perform the simultaneous homogenization and dimension reduction process in the von Kármán case and obtain that the corresponding homogenized model, i.e. the Γ-limit of $h^{-4}\mathcal{E}^h(\hat{y}^h)$ as $h \downarrow 0$, is given by

$$\mathcal{E}^0(u, v_2, v_3, w) = \mathcal{K}_{(h)}(m_d) - \int_0^L (f_2v_2 + f_3v_3)\mathrm{d}x_1,$$

where the functions u, v_2, v_3 and w are the weak limits of the scaled displacements and the twist functions, respectively, and m_d is given by (6). Moreover, the resulting limit elastic energy density (depending on a given subsequence of the diminishing thickness (h)) can be calculated according to

$$\mathcal{K}_{(h)}(m_d) = \lim_{h\downarrow 0} \int_\Omega Q^h(x, \imath(m_d) + \operatorname{sym}\nabla_h\psi_{m_d}^h)\mathrm{d}x, \tag{7}$$

where Q^h is the quadratic form approximating the energy density W^h, and $(\psi_{m_d}^h)$ the sequence (which we call the relaxation sequence) that satisfies certain minimality property (see (21), (22) below). Confer Section 2.5 for more details.

As we already stressed out, our aim is to study the stationary points of the energy functional $\mathcal{E}^h$ rather than just global minimizers attainable through the Γ-convergence techniques. The weak form of the Euler–Lagrange equation of the functional $\mathcal{E}^h$, assuming the zero boundary condition on the zero cross-section $\{0\} \times \omega$, formally reads:

$$\int_\Omega \Big(DW^h(x, \nabla_h y^h) : \nabla_h \phi - h^3 (f_2 \phi_2 + f_3 \phi_3) \Big) \, \mathrm{d}x = 0 \,, \tag{8}$$

for all test functions $\phi \in H^1_\omega(\Omega, \mathbb{R}^3) = \{\phi \in H^1(\Omega) \ : \ \phi|_{\{0\}\times\omega} = 0\}$. This notion of stationarity is the standard one, but possibly not best suited for the nonlinear elasticity. Namely, it is still an open question whether under physical growth assumptions on the energy densities W^h, global or suitably defined local minimizers of $\mathcal{E}^h$ satisfy the Euler–Lagrange equation [3]. In this paper we additionally require linear growth and continuity of the stress (cf. hypothesis H5 below). This is also done in [24, 26]. There is an alternative notion of first-order stationarity in elasticity, proposed by Ball in [3], and that concept is compatible with a physical growth condition which roughly says that the energy blows up if the deformation degenerates. While the authors in [11] managed to deal with the alternative stationarity condition and to systematically derive the corresponding stationarity conditions for the limit models, our method is not compatible with that mainly because of the possibility of interpenetration of the matter and we remain in the previously discussed setting.

Now we are in position to state the main result of the paper.

Theorem 1.1. *Let the sequence (W^h) describe an admissible composite material (see $(C1)$, $(C2)$, $(C3)$ below) and let $(y^h) \subset H^1_\omega(\Omega, \mathbb{R}^3)$ be a sequence satisfying (2). Then the sequence of deformations and sequences of scaled displacements (we take $\hat{y}^h = y^h$) converge (on a subsequence) as follows:*

$$\begin{aligned}
&y^h \to x_1 e_1 \textit{strongly in } H^1(\Omega, \mathbb{R}^3) \,, \\
&u^h \rightharpoonup u \textit{ weakly in } H^1_0(0, L) \,, \\
&v^h_i \to v_i \textit{ strongly in } H^1_0(0, L) \,, \textit{ and } v_i \in H^2_0(0, L) \textit{ for } i = 2, 3 \,, \\
&w^h \rightharpoonup w \textit{ weakly in } H^1_0(0, L) \,.
\end{aligned}$$

Let $f^h = h^3(f_2 e_2 + f_3 e_3)$ with $f_2, f_3 \in L^2(0, L)$ be an external load and assume that (y^h) are stationary points of the energy functional $\mathcal{E}^h$, i.e. solve equation (8), then (u, v_2, v_3, w) is a stationary point of the limit energy functional $\mathcal{E}^0$.

Big part of the proof of Theorem 1.1 (compactness) does not differ much from the case of materials without a microstructure, which is already available in the literature. These results are comprised and properly referenced in Theorems 2.1 and 2.4 below in Section 2. Hence, the main focus here is the statement that stationarity of the sequence of deformations y^h of the energy functional $\mathcal{E}^h$ (in the sense of (8)) implies the stationarity of the point (u, v_2, v_3, w) for the limit energy functional $\mathcal{E}^0$. The key in proving that statement are the orthogonality properties provided in Lemma 2.5 and Lemma 3.1, respectively, which essentially allows us to identify two sequences: the relaxation sequence (ψ^h_m) from (7) and the sequence of correctors (ψ^h) from (5) up to L^2-concentrations, which are irrelevant for identification of weak limits. Namely, Lemma 2.5 tells us that the orthogonality property is automatically satisfied by the relaxation sequence (ψ^h_m), while Lemma 3.1 proves this property for the sequence of correctors (ψ^h) by using the equations. The proof of Lemma 3.1, together with the proof of Theorem

1.1, and identification of limit Euler–Lagrange equations are the subject of Section 3, while some technical results can be found in the Appendix. We emphasize at this point that, up to some technical peculiarities, the same approach can be utilized for studying the convergence of stationary points of the von Kármán plate model, and the analogous result holds true.

Finally, in Section 4 we consider materials with random microstructure satisfying the von Kármán scaling and provide an explicit cell formula for the limit energy density of the functional $\mathcal{K}_{(h)}$ (cf. Proposition 2.6). This result also covers the case of materials with periodic and almost periodic microstructure.

2. Preliminaries

2.1. Notation

The set $\Omega = (0, L) \times \omega \subset \mathbb{R}^3$ is a Lipschitz domain describing the canonical configuration of a rod of length $L > 0$ and shape $\omega \subset \mathbb{R}^2$. Vectors e_1, e_2, e_3 denote the canonical basis of $\mathbb{R}^3$ and $(x_1, x') \in \mathbb{R}^3$, with $x' = (x_2, x_3) \in \mathbb{R}^2$, denote the coordinates of a point in $\mathbb{R}^3$ with respect to that basis. Also, we will frequently use the projection of a point $x \in \mathbb{R}^3$ to x'-plane, denoted by $\mathfrak{p}_{x'}(x) = (0, x')^T$. For a given thickness $h > 0$, the scaled gradient is denoted by $\nabla_h = (\partial_1, \frac{1}{h}\partial_2, \frac{1}{h}\partial_3)$. The space of real 3×3 matrices is denoted by $\mathbb{R}^{3\times3}$, while $\mathbb{R}^{3\times3}_{\text{sym}}$, $\mathbb{R}^{3\times3}_{\text{skw}}$ and $\text{SO}(3)$ denote the subspaces of symmetric, skew-symmetric, and special orthogonal matrices, respectively. For a skew-symmetric matrix A we denote its axial vector by $\text{axl}\, A = (A_{32}, A_{13}, A_{21})$. By $\iota : \mathbb{R}^3 \to \mathbb{R}^{3\times3}$ we denote the inclusion $\iota(v) = v \otimes e_1$. Depending on the context, by $|\cdot|$ we denote both the Lebesgue measure of a set and the euclidean norm of a vector in $\mathbb{R}^d$. The space of smooth functions on $[0, L]$ which are vanishing at zero will be denoted by $C_0^\infty([0, L])$, while the space of smooth functions on Ω with compact support will be denoted by $C_c^\infty(\Omega)$. Given two functions $\phi, \psi \in L^1(\Omega, \mathbb{R}^3)$, we define the *twist* function $\mathfrak{t}(\phi, \psi) : (0, L) \to \mathbb{R}$ by

$$\mathfrak{t}(\phi, \psi)(x_1) = \int_\omega (x_2\psi - x_3\phi)\mathrm{d}x' \,.$$

Finally, the moments of a function $\Psi \in L^1(\Omega, \mathbb{R}^{3\times3})$ are denoted as follows. The zeroth moment $\overline{\Psi} : (0, L) \to \mathbb{R}^{3\times3}$ is defined by

$$\overline{\Psi}(x_1) = \int_\omega \Psi(x)\mathrm{d}x' \,, \tag{9}$$

and first-order moments $\widetilde{\Psi}, \widehat{\Psi} : (0, L) \to \mathbb{R}^{3\times3}$ are defined by

$$\widetilde{\Psi}(x_1) = \int_\omega x_2\Psi(x)\mathrm{d}x' \,, \qquad \widehat{\Psi}(x_1) = \int_\omega x_3\Psi(x)\mathrm{d}x' \,. \tag{10}$$

2.2. von Kármán rod model – supplement

Let $\omega \subset \mathbb{R}^2$ be a Lipschitz domain with Lebesgue measure $|\omega| = 1$ and assume that coordinate axes are chosen such that

$$\int_\omega x_2\mathrm{d}x' = \int_\omega x_3\mathrm{d}x' = \int_\omega x_2x_3\mathrm{d}x' = 0 \,.$$

By $\Omega^h = (0, L) \times h\omega$ we denote the material domain of a rod-like body of thickness $h > 0$ and length $L > 0$. Performing the standard change of variables $\Omega_h \ni \hat{x} \mapsto x \in \Omega$, given by $x_1 = \hat{x}_1$, $x' = \frac{1}{h}\hat{x}'$, we will in the sequel work on the canonical domain $\Omega = (0, L) \times \omega$. For every $h > 0$, the (scaled) energy functional of a deformation $y^h : \Omega \to \mathbb{R}^3$ is given by expression (1).

For the elastic energy densities W^h we have more or less standard hypotheses for nonlinear composite material, which are listed in the sequel.

Nonlinear material law. Let α, β, ϱ and κ be positive constants such that $\alpha \le \beta$. The class $\mathcal{W}(\alpha, \beta, \varrho, \kappa)$ consists of all measurable functions $W : \mathbb{R}^{3\times 3} \to [0, +\infty]$ satisfying:

(H1) frame indifference: $W(RF) = W(F)$ for all $F \in \mathbb{R}^{3\times 3}$ and $R \in \mathrm{SO}(3)$;

(H2) non-degeneracy:

$$\begin{aligned} W(F) &\ge \alpha \,\mathrm{dist}^2(F, \mathrm{SO}(3)) \quad \text{for all } F \in \mathbb{R}^{3\times 3}\,, \\ W(F) &\le \beta \,\mathrm{dist}^2(F, \mathrm{SO}(3)) \quad \text{for all } F \in \mathbb{R}^{3\times 3} \text{ with } \mathrm{dist}^2(F, \mathrm{SO}(3)) \le \varrho\,; \end{aligned}$$

(H3) minimality at identity: $W(I) = 0$;

(H4) quadratic expansion at identity: $W(I + G) = Q(G) + o(|G|^2)$ as $G \to 0$ ($G \in \mathbb{R}^{3\times 3}$), where $Q : \mathbb{R}^{3\times 3} \to \mathbb{R}$ is a quadratic form;

(H5) linear stress growth: $|DW(F)| \le \kappa(|F| + 1)$ for all $F \in \mathbb{R}^{3\times 3}$.

Admissible composite material. For α, β, ϱ and κ positive constants as above, a family of functions $W^h : \Omega \times \mathbb{R}^{3\times 3} \to [0, +\infty]$ describes an admissible composite material of class $\mathcal{W}(\alpha, \beta, \varrho, \kappa)$ if the following hypotheses hold:

(C1) for every $h > 0$, W^h is almost everywhere equal to a Borel function on $\Omega \times \mathbb{R}^{3\times 3}$;

(C2) for every $h > 0$, $W^h(x, \cdot) \in \mathcal{W}(\alpha, \beta, \varrho, \kappa)$ for a.e. $x \in \Omega$;

(C3) there exists a monotone function $r : [0, +\infty) \to [0, +\infty)$ such that $r(\delta) \downarrow 0$ as $\delta \downarrow 0$ and

$$\forall G \in \mathbb{R}^{3\times 3}\,,\ \forall h > 0 \ : \ \operatorname*{ess\,sup}_{x\in\Omega} |W^h(x, I + G) - Q^h(x, G)| \le r(|G|)|G|^2\,, \tag{11}$$

where $Q^h(x, \cdot)$ are quadratic forms defined in (H4).

The given quadratic form $Q^h(x, \cdot)$ can be (uniquely) represented by a positive semidefinite linear operator $\mathbb{A}^h(x)$, i.e.

$$Q^h(x, F) = \frac{1}{2}\mathbb{A}^h(x)F : F\,, \quad \text{for all } F \in \mathbb{R}^{3\times 3} \text{ and for a.e. } x \in \Omega\,.$$

Assuming that Q^h corresponds to an elastic energy density W^h belonging to a family of elastic energy densities describing an admissible composite material of the class $\mathcal{W}(\alpha, \beta, \varrho, \kappa)$, one can easily prove that Q^h is a Carathéodory function which satisfies:

(a) $\alpha|\operatorname{sym} F|^2 \le Q^h(x, F) = Q^h(x, \operatorname{sym} F) \le \beta|\operatorname{sym} F|^2$, for all $F \in \mathbb{R}^{3\times 3}$;

(b) $|Q^h(x, F_1) - Q^h(x, F_2)| \le \beta|\operatorname{sym} F_1 - \operatorname{sym} F_2||\operatorname{sym} F_1 + \operatorname{sym} F_2|$, for all $F_1, F_2 \in \mathbb{R}^{3\times 3}$.

2.3. Rigidity and compactness

Using the theorem of geometric rigidity [14], the following result has been established in [23].

Theorem 2.1. *Let $(y^h) \subset H^1(\Omega, \mathbb{R}^3)$ be a sequence satisfying*

$$\limsup_{h \downarrow 0} \frac{1}{h^4} \int_\Omega \operatorname{dist}^2(\nabla_h y^h, \mathrm{SO}(3)) \mathrm{d}x < +\infty \,.$$

Then there exist: a sequence of maps $(R^h) \subset C^\infty([0,L], \mathrm{SO}(3))$, a sequence of constant rotations $(\bar{R}^h) \subset \mathrm{SO}(3)$ and constants $(c^h) \subset \mathbb{R}^3$ such that the sequence $(\hat{y}^h)$, defined by $\hat{y}^h = (\bar{R}^h)^T y^h - c^h$, satisfies

$$\begin{aligned} \|\nabla_h \hat{y}^h - R^h\|_{L^2(\Omega)} &\leq Ch^2 \,, \\ \|(R^h)'\|_{L^2(0,L)} &\leq Ch \,, \\ \|R^h - I\|_{L^2(0,L)} &\leq Ch \,. \end{aligned} \tag{12}$$

The sequence of constants (c^h) in the previous theorem can be chosen such that

$$\int_\Omega (\hat{y}_1^h - x_1) \mathrm{d}x = 0 \,, \qquad \int_\Omega \hat{y}_i^h \mathrm{d}x = 0 \quad \text{for } i = 2, 3 \,.$$

Next, we introduce the following ansatz for $(\hat{y}^h)$:

$$\begin{aligned} \hat{y}_1^h &= x_1 + h^2 \left(u^h - x_2 \frac{R_{21}^h}{h} - x_3 \frac{R_{31}^h}{h} \right) + h^2 \beta_1^h \,, \\ \hat{y}_i^h &= hx_i + hv_i^h + h^2 w^h x_i^\perp + h^2 \beta_i^h \,, \quad \text{for } i = 2, 3 \,, \end{aligned} \tag{13}$$

where $x^\perp = (0, -x_3, x_2)$, and functions u^h, v_2^h, v_3^h, and w^h are defined in (3) and (4).

Remark 2.2. *Using* (12)*, the Poincaré inequality, the fact that $y^h(0, x_2, x_3) = (0, hx_2, hx_3)$, and the construction from [23], it can be shown that $|R^h(0) - I| \leq Ch^{3/2}$ for some $C > 0$. Thus the boundary condition imply that $\bar{R}^h$ can be taken to be equal to identity matrix and c^h can be taken to be zero (i.e., we can take $\hat{y}^h = y^h$).*

Remark 2.3. *Observe that the proposed ansatz is a slight modification of the ansatz for the same sequence $(\hat{y}^h)$ from [23, Theorem 2.2 (f)]. In lieu of terms $(v_i^h)'$, $i = 2, 3$, we set $\frac{1}{h} R_{i1}^h$, respectively. This enables us to control the full scaled gradient of the corrector sequence (β^h) in the L^2-norm (see Theorem 2.4 below), which is crucial for application of our method in the analysis afterwards.*

Theorem 2.4. *Let the assumption and notation of the previous theorem be retained and let $y^h(0, x_2, x_3) = (0, hx_2, hx_3)$. For sequences (u^h), (v_i^h), $i = 2, 3$, and (w^h) defined above, we have the following convergence results which hold on a subsequence:*

$$\begin{aligned} &u^h \rightharpoonup u \text{ weakly in } H_0^1(0, L) \,, \\ &v_i^h \to v_i \text{ strongly in } H_0^1(0, L) \,, \text{ and } v_i \in H_0^2(0, L) \text{ for } i = 2, 3 \,, \\ &w^h \rightharpoonup w \text{ weakly in } H_0^1(0, L) \,. \end{aligned}$$

Moreover, the sequence of corrector functions (β^h) satisfies the uniform bounds: $\|\beta^h\|_{L^2(\Omega)} \leq Ch$ and $\|\nabla_h \beta^h\|_{L^2(\Omega)} \leq C$.

Proof. The proof follows the lines of the proof of Theorem 2.2 from [23], but we include it here for the reader's convenience. From Remark 2.2 we conclude that we can take $\hat{y}^h = y^h$. Let us define

$$A^h := \frac{1}{h}(R^h - I)\,.$$

From the previous theorem we have $\|R^h - I\|_{L^2(0,L)} \leq Ch$ and $\|(R^h)'\|_{L^2(0,L)} \leq Ch$, which implies the uniform bound $\|A^h\|_{H^1(0,L)} \leq C$. Therefore, (up to a subsequence) $A^h \rightharpoonup A$ weakly in $H^1((0,L),\mathbb{R}^{3\times 3})$. From the compactness of the Sobolev embedding $H^1((0,L),\mathbb{R}^{3\times 3}) \hookrightarrow L^\infty((0,L),\mathbb{R}^{3\times 3})$, we conclude $A^h \to A$ strongly in $L^\infty((0,L),\mathbb{R}^{3\times 3})$. Direct calculation reveals the identities

$$A^h + (A^h)^T = -hA^h(A^h)^T \quad \text{and} \quad \frac{1}{h^2}\operatorname{sym}(R^h - I) = \frac{1}{2h}(A^h + (A^h)^T)\,,$$

which respectively imply $A^T = -A$ and

$$\frac{1}{h^2}\operatorname{sym}(R^h - I) \to \frac{1}{2}A^2 \quad \text{strongly in } L^\infty((0,L),\mathbb{R}^{3\times 3})\,. \tag{14}$$

Since $\|\nabla_h y^h - R^h\|_{L^2(\Omega)} \leq Ch^2$, using the triangle inequality and established convergence results, we conclude

$$\frac{1}{h}(\nabla_h y^h - I) \to A \quad \text{strongly in } L^2(\Omega,\mathbb{R}^{3\times 3})\,. \tag{15}$$

By construction we have $\int_0^L u^h(x_1)\mathrm{d}x_1 = 0$. Thus, the Poincaré and Jensen inequalities together with (14) imply

$$\begin{aligned}\|u^h\|_{L^2(0,L)} &\leq C_P\|(u^h)'\|_{L^2(0,L)} \leq \frac{C_P}{h^2}\|\partial_1 y_1^h - 1\|_{L^2(\Omega)} \\ &\leq \frac{C_P}{h^2}\|\partial_1 y_1^h - R_{11}^h\|_{L^2(\Omega)} + \frac{C_P}{h^2}\|R_{11}^h - 1\|_{L^2(\Omega)} \leq C\,.\end{aligned}$$

Therefore, up to a subsequence $u^h \rightharpoonup u$ weakly in $H^1(0,L)$. Similarly, $\int_0^L v_i^h(x_1)\mathrm{d}x_1 = 0$ for $i = 2,3$, and

$$\|(v_i^h)'\|_{L^2(0,L)} \leq \frac{1}{h}\|\partial_1 y_i^h\|_{L^2(\Omega)} \leq C\,.$$

Hence, (up to a subsequence) $v_i^h \rightharpoonup v_i$ weakly in $H^1(0,L)$. Moreover, since

$$(v_i^h)' = \int_\omega \frac{\partial_1 y_i^h}{h}\mathrm{d}x' \to A_{i1} \quad \text{strongly in } L^2(\Omega,\mathbb{R}^{3\times 3})\,,$$

one concludes that $A_{i1} = v_i'$ for $i = 2,3$. Since $A_{i1} \in H^1(0,L)$, we conclude $v_i \in H^2(0,L)$ for $i = 2,3$. Next, we consider the sequence of twist functions (w^h). Note that they can be written as

$$\begin{aligned}w^h(x_1) = &\frac{1}{\mu(\omega)}\int_\omega x_2\left(\frac{h^{-1}y_3^h - x_3}{h} - \frac{1}{h^2}\int_\omega y_3^h\mathrm{d}x'\right)\mathrm{d}x' \\ &- \frac{1}{\mu(\omega)}\int_\omega x_3\left(\frac{h^{-1}y_2^h - x_2}{h} - \frac{1}{h^2}\int_\omega y_2^h\mathrm{d}x'\right)\mathrm{d}x'\,.\end{aligned}$$

For the above integrands we have (according to (15) and the Poincaré inequality):

$$\frac{h^{-1}y_3^h - x_3}{h} - \frac{1}{h^2}\int_\omega y_3^h \mathrm{d}x' \to A_{32}x_2 \quad \text{strongly in } L^2(\Omega)\,;$$
$$\frac{h^{-1}y_2^h - x_2}{h} - \frac{1}{h^2}\int_\omega y_2^h \mathrm{d}x' \to -A_{32}x_3 \quad \text{strongly in } L^2(\Omega)\,.$$

Therefore, w^h converges strongly in the L^2-norm to the function $w = A_{32} \in L^2(0,L)$. Using the a priori estimate $\|\nabla_h y^h - R^h\|_{L^2} \leq Ch^2$ and the normality of rotation matrix columns, we conclude the uniform bound $\|(w^h)'\|_{L^2(0,L)} \leq C$. Hence, $w^h \rightharpoonup w$ weakly in the H^1-norm. Observe that the limit matrix $A \in H^1((0,L), \mathbb{R}^{3\times 3})$ is completely identified by the limits $u, w \in H^1(0,L)$ and $v_1, v_2 \in H^2(0,L)$ in the following way

$$A = \begin{pmatrix} 0 & -v_2' & -v_3' \\ v_2' & 0 & -w \\ v_3' & w & 0 \end{pmatrix}. \tag{16}$$

Finally, we consider the sequence of corrector functions (β^h) given by:

$$\beta_1^h(x) = \frac{y_1^h(x) - x_1}{h^2} - u^h(x_1) + x_2\frac{R_{21}^h(x_1)}{h} + x_3\frac{R_{31}^h(x_1)}{h}\,,$$
$$\beta_i^h(x) = \frac{1}{h}\left(\frac{y_i^h(x) - hx_i}{h} - v_i^h(x_1) - hw^h(x_1)x_i^\perp\right), \quad i = 2,3\,.$$

For brevity reasons, let us denote $\partial_i^h = \frac{1}{h}\partial_i$, then for $i = 2,3$ we compute

$$\partial_i\beta_1^h = \frac{1}{h^2}\partial_i y_1^h + \frac{R_{i1}^h}{h} = \frac{1}{h}\left(\partial_i^h y_1^h - R_{1i}^h\right) + \frac{R_{i1}^h + R_{1i}^h}{h}\,.$$

The first term on the right-hand side is bounded in the L^2-norm due to $\|\nabla_h y^h - R^h\|_{L^2(\Omega)} \leq Ch^2$, and the second one due to (14). Thus, $\|\partial_i\beta_1^h\|_{L^2(\Omega)} \leq Ch$ for $i = 2,3$. Since $\int_\omega \beta_1^h(x)\mathrm{d}x' = 0$, using the Poincaré inequality we conclude

$$\|\beta_1^h(x_1,\cdot)\|_{L^2(\omega)}^2 \leq C\left(\|\partial_2\beta_1^h(x_1,\cdot)\|_{L^2(\omega)}^2 + \|\partial_3\beta_1^h(x_1,\cdot)\|_{L^2(\omega)}^2\right) \quad \text{for a.e. } x_1 \in (0,L)\,.$$

Integrating the latter inequality along x_1-direction yields the $L^2(\Omega)$-bound on β_1^h of order $O(h)$. The identity

$$\partial_1\beta_1^h = \frac{\partial_1 y_1^h - 1}{h^2} - (u^h)' + x_2\frac{(R_{21}^h)'}{h} + x_3\frac{(R_{31}^h)'}{h}\,,$$

directly implies the uniform bound $\|\partial_1\beta_1^h\|_{L^2(\Omega)} \leq C$. Straightforward calculations reveal

$$\partial_j\beta_i^h = \frac{1}{h}\left(\partial_j^h y_i^h - \delta_{ij} - (-1)^j(1-\delta_{ij})hw^h\right), \quad \text{for } i,j = 2,3\,,$$

where we have used $\partial_j x_i^\perp = (-1)^j(1-\delta_{ij})$. Furthermore,

$$(\operatorname{sym}\nabla\beta^h)_{ij} = \frac{\partial_j\beta_i^h + \partial_i\beta_j^h}{2} = \frac{1}{h}\left(\operatorname{sym}(\nabla_h y^h - I)\right)_{ij}\,, \quad \text{for } i,j = 2,3\,,$$

which implies the uniform bound $\|(\mathrm{sym}\,\nabla\beta^h)_{ij}\|_{L^2(\Omega)} \leq Ch$ for $i,j = 2,3$. Note that for a.e. $x_1 \in (0,L)$ the function $(\beta_2^h(x_1,\cdot), \beta_3^h(x_1,\cdot))$ belongs to the closed subspace

$$\mathcal{B} = \left\{ \alpha \in H^1(\omega,\mathbb{R}^2) \;:\; \int_\omega \alpha(x')\mathrm{d}x' = 0\,, \int_\omega (x_3\alpha_2 - x_2\alpha_3)\mathrm{d}x' = 0 \right\},$$

on which a Korn type inequality [30] holds

$$\|\beta_2^h(x_1,\cdot)\|^2_{H^1(\omega)} + \|\beta_3^h(x_1,\cdot)\|^2_{H^1(\omega)} \leq C \sum_{i,j=2,3} \|(\mathrm{sym}\,\nabla\beta^h(x_1,\cdot))_{ij}\|^2_{L^2(\omega)}\,.$$

Integrating the latter with respect to x_1, yields the respective uniform $H^1(\Omega)$-bound. Hence, we proved $\|\beta^h\|_{L^2(\Omega)} \leq Ch$. Finally,

$$\begin{aligned}\partial_1\beta_i^h &= \frac{1}{h}\left(\frac{\partial_1 y_i^h}{h} - (v_i^h)' - h(w^h)'x_i^\perp\right)\\ &= \frac{1}{h}\left(\frac{\partial_1 y_i^h - R_{1i}^h}{h} - \frac{1}{h}\int_\omega (\partial_1 y_i^h - R_{1i}^h)\mathrm{d}x' - h(w^h)'x_i^\perp\right), \quad \text{for } i,j = 2,3\,,\end{aligned}$$

and the previously established convergence results imply $\|\partial_1\beta_i^h\|_{L^2(\Omega)} \leq C$. Thus, we have proved $\|\nabla_h\beta^h\|_{L^2(\Omega)} \leq C$. The boundary conditions for w^h, v_i^h, u^h, v_i, u follow from the boundary condition for y^h, Remark 2.2 and the above convergence results. □

2.4. Strain and stress estimates

For every $h > 0$, using the rotation matrix function R^h, the strain tensor G^h is implicitly defined through the following decomposition of the scaled deformation gradient

$$\nabla_h y^h = R^h(I + h^2 G^h)\,.$$

The explicit identity $G^h = h^{-2}(R^h)^T(\nabla_h y^h - R^h)$ directly implies with (12) the L^2-uniform bound on the sequence (G^h). Hence, there exists $G \in L^2(\Omega,\mathbb{R}^{3\times3})$ such that $G^h \rightharpoonup G$ on a subsequence. Our aim is to describe the symmetrized strain $\mathrm{sym}\,G^h$ in more detail. First, we explicitly involve the limit functions $u, w \in H^1(0,L)$ and $v_1, v_2 \in H^2(0,L)$ into our ansatz (13) in the following way:

$$\begin{aligned}\frac{y_1^h - x_1}{h^2} &= u - x_2 v_2' - x_3 v_3' + \psi_1^h\,,\\ \frac{y_i^h - hx_i}{h^2} &= \frac{v_i}{h} + w x_i^\perp + \psi_i^h\,, \quad \text{for } i = 2,3\,,\end{aligned}$$

where

$$\begin{aligned}\psi_1^h &= u^h - u - x_2\Big(\frac{R_{21}^h}{h} - v_2'\Big) - x_3\Big(\frac{R_{31}^h}{h} - v_3'\Big) + \beta_1^h\,,\\ \psi_i^h &= \frac{1}{h}(v_i^h - v_i) + (w^h - w)x_i^\perp + \beta_i^h\,, \quad \text{for } i = 2,3\,.\end{aligned}$$

Previously established convergence results imply that $(\psi_1^h, h\psi_2^h, h\psi_3^h) \to 0$ strongly in the L^2-norm. Moreover, the derivatives are given by

$$\begin{aligned}
\partial_1 \psi_1^h &= (u^h)' - u' - x_2\Big(\frac{(R_{21}^h)'}{h} - v_2''\Big) - x_3\Big(\frac{(R_{31}^h)'}{h} - v_3''\Big) + \partial_1 \beta_1^h \,, \\
\partial_j^h \psi_1^h &= \frac{v_j'}{h} - \frac{R_{j1}^h}{h^2} + \partial_j^h \beta_1^h \,, \quad \text{for } j = 2,3 \,, \\
\partial_j^h \psi_i^h &= \frac{(-1)^j}{h}(1-\delta_{ij})(w^h - w) + \partial_j^h \beta_i^h \,, \quad \text{for } i,j = 2,3 \,, \\
\partial_1 \psi_i^h &= \frac{1}{h}\Big((v_i^h)' - v_i'\Big) + \Big((w^h)' - w'\Big)\, x_i^\perp + \partial_1 \beta_i^h \,, \quad \text{for } i = 2,3 \,,
\end{aligned}$$

which together with known convergence results immediately gives $\| \operatorname{sym} \nabla_h \psi^h \|_{L^2(\Omega)} \leq C$. Invoking (16), we obtain the following representation:

$$\frac{1}{h^2} \operatorname{sym}\Big(\nabla_h y^h - I\Big) = u' e_1 \otimes e_1 + \operatorname{sym}(\imath(A' \mathfrak{p}_{x'})) + \operatorname{sym} \nabla_h \psi^h \,, \tag{17}$$

Additionally, using $(\beta_2^h(x_1,\cdot), \beta_3^h(x_1,\cdot)) \in \mathcal{B}$ for a.e. $x_1 \in (0,L)$, one can easily check that

$$\int_\omega (x_3 \psi_2^h - x_2 \psi_3^h) \mathrm{d}x' = -(w^h - w) \int_\omega (x_2^2 + x_3^2) \mathrm{d}x' \to 0 \quad \text{strongly in } L^2 \,.$$

Next, we compute the symmetrized strain using decomposition (17):

$$\begin{aligned}
\operatorname{sym} G^h &= \frac{1}{h^2} \operatorname{sym}\Big((R^h)^T \nabla_h y^h - I\Big) = \frac{1}{h^2} \operatorname{sym}((R^h - I)^T \nabla_h y^h) + \frac{1}{h^2} \operatorname{sym}(\nabla_h y^h - I) \\
&= \frac{1}{h^2} \operatorname{sym}((R^h - I)^T (\nabla_h y^h - R^h)) - \frac{1}{h^2} \operatorname{sym}(R^h - I) + \frac{1}{h^2} \operatorname{sym}(\nabla_h y^h - I) \\
&=: \tilde{o}^h - \frac{1}{h^2} \operatorname{sym}(R^h - I) + u' e_1 \otimes e_1 + \operatorname{sym}(\imath(A' \mathfrak{p}_{x'})) + \operatorname{sym} \nabla_h \psi^h \\
&= u' e_1 \otimes e_1 + \operatorname{sym}(\imath(A' \mathfrak{p}_{x'})) - \frac{1}{2} A^2 + \operatorname{sym} \nabla_h \psi^h + o^h \\
&=: \operatorname{sym} H + \operatorname{sym} \nabla_h \psi^h + o^h \,,
\end{aligned}$$

where $\tilde{o}^h, o^h \to 0$ strongly in $L^2(\Omega, \mathbb{R}^{3\times 3})$, and $\operatorname{sym} H = u' e_1 \otimes e_1 + \operatorname{sym}(\imath(A' \mathfrak{p}_{x'})) - \frac{1}{2} A^2$. In this way we decomposed $\operatorname{sym} G^h$ into a fixed and a corrector part. A part of $\operatorname{sym} H$ can be further transferred to the corrector terms as follows:

$$\begin{aligned}
\operatorname{sym} H &= \Big(u' + \frac{1}{2}\Big((v_2')^2 + (v_3')^2\Big)\Big) e_1 \otimes e_1 + \operatorname{sym}(\imath(A' \mathfrak{p}_{x'})) \\
&\quad + \frac{1}{2} \begin{pmatrix} 0 & v_3' w & -v_2' w \\ v_3' w & w^2 + (v_2')^2 & v_2' v_3' \\ -v_2' w & v_2' v_3' & w^2 + (v_3')^2 \end{pmatrix} \\
&=: \operatorname{sym}(\imath(m_d)) + \operatorname{sym} \nabla_h \alpha^h - \operatorname{sym} \imath(\partial_1 \alpha^h) \,,
\end{aligned}$$

where

$$m_d = \Big(u' + \frac{1}{2}((v_2')^2 + (v_3')^2)\Big) e_1 + A' \mathfrak{p}_{x'} \,, \tag{18}$$

and

$$\alpha^h(x) = h \begin{pmatrix} x_2 v_3' w - x_3 v_2' w \\ \frac{1}{2}x_2(w^2 + (v_2')^2) + \frac{1}{2}x_3 v_2' v_3' \\ \frac{1}{2}x_2 v_2' v_3' + \frac{1}{2}x_3(w^2 + (v_3')^2) \end{pmatrix} .$$

Finally, we have decomposition

$$\operatorname{sym} G^h = \operatorname{sym}(\imath(m_d)) + \operatorname{sym} \nabla_h \psi^h + o^h \,, \tag{19}$$

with the updated corrector sequence ψ^h (by adding the function α^h to the original ψ^h) and the L^2-zero convergent part o^h.

The stress field $E^h : \Omega \to \mathbb{R}^{3\times 3}$ is defined by

$$E^h := \frac{1}{h^2} DW^h(\cdot, I + h^2 G^h) \,.$$

The assumption (C3) on W^h, in particular estimate (11), implies that W^h is differentiable a.e. in $x \in \Omega$ and

$$\forall\, G \in \mathbb{R}^{3\times 3} \,,\ \forall\, h > 0 \ : \ \operatorname*{ess\,sup}_{x\in\Omega} |DW^h(x, I + G) - \mathbb{A}^h(x) G| \le r(|G|)|G| \,,$$

and therefore (see the property (a) of Q^h),

$$|DW^h(\cdot, I + h^2 G^h)| \le r(h^2|G^h|) h^2 |G^h| + \beta h^2 |G^h|, \quad \text{a.e. in } \Omega \,.$$

Let us denote the set

$$B_h := \{x \in \Omega \ : \ h^2 |G^h(x)| \le 1\} \,,$$

then from the previous inequality

$$|DW^h(\cdot, I + h^2 G^h)| \le C h^2 |G^h| \quad \text{pointwise in } B_h \,,$$

which yields

$$|E^h| \le C|G^h| \quad \text{pointwise in } B_h \,.$$

On the other hand on $\Omega \backslash B_h$, i.e. on the set where $|G^h| > h^{-2}$ a.e., applying hypothesis (H5) we conclude

$$|E^h| \le \frac{\kappa}{h^2}\left(|I + h^2 G^h| + 1\right) \le \frac{\kappa}{h^2}\left(h^2 |G^h| + \sqrt{3} + 1\right) \le C|G^h| \quad \text{pointwise in } \Omega \backslash B_h \,.$$

Therefore, we have a uniform estimate on the whole set,

$$|E^h| \le C|G^h| \quad \text{pointwise in } \Omega \,, \tag{20}$$

which together with the uniform L^2-bound for the strain sequence (G^h) implies the uniform L^2-bound on (E^h) and consequently the weak convergence (on a subsequence)

$$E^h \rightharpoonup E \quad \text{in } L^2(\Omega, \mathbb{R}^{3\times 3}) \,.$$

2.5. Representation of elastic energy functionals

In this subsection we briefly recall a variational approach for general (non-periodic) simultaneous homogenization and dimension reduction in the framework of three-dimensional nonlinear elasticity theory. This approach has been thoroughly undertaken in case of von Kármán plate [32] and bending rod [22], while the linear plate model has been outlined in [8]. The theorem on geometric rigidity provides a decomposition of the symmetrized strain to a sum of a fixed and a corrector part (cf. previous section). Utilizing the corresponding Griso's decomposition [17, 18] gives a further characterization of the corrector part, which enables an operational representation of the elastic energies (cf. Lemma 2.5 below), suitable for the application of appropriate Γ-convergence techniques to eventually identify the limiting elastic energy.

In the following we only provide basic steps of the method and state the final results. To start with, let us define so called lower and upper Γ-limits. For a monotonically decreasing to zero sequence of positive numbers $(h) \subset (0,+\infty)$, $m \in L^2(\Omega,\mathbb{R}^3)$ and an open set $O \subset (0,L)$, we define:

$$\mathcal{K}^-_{(h)}(m,O) = \inf\Big\{ \liminf_{h\downarrow 0} \int_{O\times\omega} Q^h(x, \operatorname{sym}\imath(m) + \operatorname{sym}\nabla_h\psi^h)\mathrm{d}x \mid (\psi^h_1, h\psi^h_2, h\psi^h_3) \to 0 \text{ in } L^2(O\times\omega,\mathbb{R}^3)\,,\ \mathfrak{t}(\psi^h_2,\psi^h_3) \to 0 \text{ in } L^2(O) \Big\}\,;$$

$$\mathcal{K}^+_{(h)}(m,O) = \inf\Big\{ \limsup_{h\downarrow 0} \int_{O\times\omega} Q^h(x, \operatorname{sym}\imath(m) + \operatorname{sym}\nabla_h\psi^h)\mathrm{d}x \mid (\psi^h_1, h\psi^h_2, h\psi^h_3) \to 0 \text{ in } L^2(O\times\omega,\mathbb{R}^3)\,,\ \mathfrak{t}(\psi^h_2,\psi^h_3) \to 0 \text{ in } L^2(O) \Big\}\,.$$

The above infimization is taken over all sequences $(\psi^h) \subset H^1(O\times\omega,\mathbb{R}^3)$ such that $(\psi^h_1, h\psi^h_2, h\psi^h_3) \to 0$ and twist functions $\mathfrak{t}(\psi^h_2,\psi^h_3) \to 0$ strongly in the L^2-topology as $h \to 0$. The identical proof to the one presented for Lemma 3.4 in [32] gives the continuity of $\mathcal{K}^-_{(h)}$ and $\mathcal{K}^+_{(h)}$ with respect to the first variable. Utilizing a diagonal procedure yields the equality of $\mathcal{K}^-_{(h)}$ and $\mathcal{K}^+_{(h)}$ for a subsequence, still denoted by (h), on $L^2(\Omega,\mathbb{R}^3)\times\mathcal{O}$, where $\mathcal{O}$ denotes a family of open subsets of $(0,L)$. More precisly this is done by choosing a countable dense subset of $L^2(\Omega,\mathbb{R}^3)$ and a countable dense family of open subsets of Ω and then using the continuity property (see [8] for details). This asserts the definition of the functional

$$\mathcal{K}_{(h)}(m,O) := \mathcal{K}^-_{(h)}(m,O) = \mathcal{K}^+_{(h)}(m,O)\,, \quad \forall\, m \in L^2(\Omega,\mathbb{R}^3)\,,\ \ \forall\, O \in \mathcal{O}\,. \tag{21}$$

Adopting the strategy developed in [22, cf. Lemma 2.9 and Lemma 2.10] and [32, cf. Lemma 3.8] one can prove the following key lemma. The proof is given in [8, Lemma 2.1] and the orthogonality property is proved in [22, the proof of (III) in Lemma 2.9].

Lemma 2.5. *Let $(h) \subset (0,+\infty)$, $h \downarrow 0$, be a sequence of positive numbers which satisfies (21) for every open set $O \subset (0,L)$. Then there exists a subsequence, still denoted by (h), which satisfies that for every $m \in L^2(\Omega,\mathbb{R}^3)$ there exists $(\psi^h_m) \subset H^1(\Omega,\mathbb{R}^3)$ such that for every open subset $O \subset (0,L)$, we have*

$$\mathcal{K}_{(h)}(m,O) = \lim_{h\downarrow 0} \int_{O\times\omega} Q^h(x, \operatorname{sym}\imath(m) + \operatorname{sym}\nabla_h\psi^h_m)\mathrm{d}x\,, \tag{22}$$

and the following properties hold:

(a) $(\psi^h_{m,1}, h\psi^h_{m,2}, h\psi^h_{m,3}) \to 0$ *and* $\mathfrak{t}(\psi^h_{m,2}, \psi^h_{m,3}) \to 0$ *strongly in the* L^2*-norm as* $h \downarrow 0$.

(b) *The sequence* $(|\operatorname{sym}\nabla_h \psi^h_m|^2)$ *is equi-integrable and there exist sequences* $(\Psi^h_m) \subset H^1((0,L),\mathbb{R}^{3\times 3}_{\rm skw})$ *and* $(\vartheta^h_m) \subset H^1(\Omega,\mathbb{R}^3)$ *satisfying:* $\Psi^h_m \to 0$, $\vartheta^h_m \to 0$ *strongly in the* L^2*-norm, and*

$$\operatorname{sym}\nabla_h {\psi_m}^h = \operatorname{sym}\imath((\Psi^h_m)'\mathfrak{p}_{x'}) + \operatorname{sym}\nabla_h \vartheta^h_m \,.$$

Moreover, $(|(\Psi^h_m)'|^2)$ *and* $(|\nabla_h \vartheta^h_m|^2)$ *are equi-integrable (on a subsequence) and the following inequality holds*

$$\limsup_{h\downarrow 0}\Big(\|\Psi^h_m\|_{H^1(O)} + \|\nabla_h \vartheta^h_m\|_{L^2(O\times\omega)}\Big) \le C(\beta\|m\|^2_{L^2(O\times\omega)} + 1)\,,$$

for some $C>0$ *independent of* $O \subset (0,L)$.

(c) (orthogonality) *If* $(\varphi^h) \subset H^1(\Omega,\mathbb{R}^3)$ *is any other sequence that satisfies (a) and* $(\operatorname{sym}\nabla_h\varphi^h)$ *is bounded in* $L^2(\Omega,\mathbb{R}^{3\times 3})$*, then*

$$\lim_{h\downarrow 0}\int_\Omega \mathbb{A}^h(\operatorname{sym}\imath(m) + \operatorname{sym}\nabla_h\psi^h_m) : \operatorname{sym}\nabla_h\varphi^h \mathrm{d}x = 0\,. \tag{23}$$

(d) (uniqueness) *If* $(\varphi^h) \subset H^1(\Omega,\mathbb{R}^3)$ *is any other sequence that satisfies (22) and (a), then*

$$\|\operatorname{sym}\nabla_h\psi^h_m - \operatorname{sym}\nabla_h\varphi^h\|_{L^2(\Omega)} \to 0\,,$$

and $(|\operatorname{sym}\nabla_h\varphi^h|^2)$ *is equi-integrable.*

An important feature of the method is the localization property of *the relaxation sequence* (ψ^h_m), i.e. if we know the relaxation sequence for the interval $(0,L)$, the relaxation sequence for an arbitrary open subset $O \subset (0,L)$ and fixed $m \in L^2(\Omega,\mathbb{R}^3)$, is simply obtained by restriction. This follows from formula (22).

Finally, we provide the integral representation of the functional $\mathcal{K}_{(h)}$ (cf. [22, Proposition 2.12]). Recall from (18) that m_d is of the form $m_d = (u' + \frac{1}{2}((v_2')^2 + (v_3')^2))e_1 + A'\mathfrak{p}_{x'}$. Therefore, we consider the mapping $m : L^2(0,L) \times L^2((0,L),\mathbb{R}^{3\times3}_{\rm skw}) \to L^2(\Omega,\mathbb{R}^3)$ defined by $m(\varrho,\Psi) = \varrho e_1 + \Psi\mathfrak{p}_{x'}$.

Proposition 2.6. *Let* $(h) \subset (0,+\infty)$ *be a sequence monotonically decreasing to zero. Then there exists a subsequence, still denoted by* (h)*, and a measurable function* $Q^0 : (0,L)\times\mathbb{R}\times\mathbb{R}^3 \to \mathbb{R}$ *depending on* (h)*, such that for every open subset* $O \subset (0,L)$ *and every* $(\varrho,\Psi) \in L^2(0,L)\times L^2((0,L),\mathbb{R}^{3\times3}_{\rm skw})$ *we have*

$$\mathcal{K}_{(h)}(m(\varrho,\Psi),O) = \int_O Q^0(x_1,\varrho(x_1),\operatorname{axl}\Psi(x_1))\mathrm{d}x_1\,. \tag{24}$$

Moreover, for a.e. $x_1 \in (0,L)$, $Q^0(x_1,\cdot,\cdot) : \mathbb{R}^4 \to \mathbb{R}$ *is a bounded and coercive quadratic form.*

At this point we also define function $Q^0_1 : (0,L)\times\mathbb{R}^3 \to \mathbb{R}$ by

$$Q^0_1(x_1,v) = \min_{z\in\mathbb{R}} Q^0(x_1,z,v) \quad \text{for all } v\in\mathbb{R}^3 \text{ and a.e. } x_1 \in (0,L)\,,$$

and function $\varrho_0 : (0,L)\times\mathbb{R}^3 \to \mathbb{R}$ satisfying $Q^0_1(x_1,\operatorname{axl}F) = Q^0(x_1,\varrho_0(x_1,\operatorname{axl}F),\operatorname{axl}F)$ for all $F \in \mathbb{R}^{3\times3}_{\rm skw}$ and a.e. $x_1 \in (0,L)$. One can also prove that $Q^0_1(x_1,\cdot)$ is a bounded and coercive quadratic form for a.e. $x_1 \in (0,L)$. The linear operators associated with the quadratic forms $Q^0(x_1,\cdot,\cdot)$ and $Q^0(x_1,\cdot)$ are denoted by $\mathbb{A}^0(x_1)$ and $\mathbb{A}^0_1(x_1)$, respectively.

2.6. Variational derivative of the limit elastic energy

Let $(h) \subset (0, +\infty)$ be a monotonically decreasing to zero sequence and let $m \in L^2(\Omega, \mathbb{R}^3)$ be given. According to Lemma 2.5, there exist a subsequence still denoted by (h) and a relaxation sequence $(\psi_m^h) \subset H^1(\Omega, \mathbb{R}^3)$, depending on m, satisfying $(\psi_{m,1}^h, h\psi_{m,2}^h, h\psi_{m,3}^h) \to 0$ and $\mathfrak{t}(\psi_{m,2}^h, \psi_{m,3}^h) \to 0$ strongly in the L^2-norm, such that the limit elastic energy $\mathcal{K}_{(h)}(m) := \mathcal{K}_{(h)}(m, (0, L))$ is given by

$$\begin{aligned}\mathcal{K}_{(h)}(m) &= \lim_{h \downarrow 0} \int_\Omega Q^h(x, \operatorname{sym} \imath(m) + \operatorname{sym} \nabla_h \psi_m^h) \mathrm{d}x \\ &= \lim_{h \downarrow 0} \frac{1}{2} \int_\Omega \mathbb{A}^h(\operatorname{sym} \imath(m) + \operatorname{sym} \nabla_h \psi_m^h) : (\operatorname{sym} \imath(m) + \operatorname{sym} \nabla_h \psi_m^h) \mathrm{d}x\,.\end{aligned}$$

In the following we compute the variational derivative of $\mathcal{K}_{(h)}$ at the point m. Let $n \in L^2(\Omega, \mathbb{R}^3)$ be a test function. Then, by definition

$$\frac{\delta \mathcal{K}(m)}{\delta m}[n] = \lim_{\varepsilon \downarrow 0} \frac{\mathcal{K}(m + \varepsilon n) - \mathcal{K}(m)}{\varepsilon}\,. \tag{25}$$

With a trick of successive adding of the corresponding relaxation sequences and using the orthogonality property (23), for a suitable subsequence of (h) we calculate:

$$\begin{aligned}&\mathcal{K}_{(h)}(m + \varepsilon n) - \mathcal{K}_{(h)}(m) \\ &\quad= \lim_{h \downarrow 0} \frac{1}{2} \int_\Omega \mathbb{A}^h(\operatorname{sym} \imath(m + \varepsilon n) + \operatorname{sym} \nabla_h \psi_{m+\varepsilon n}^h) : (\operatorname{sym} \imath(m + \varepsilon n) + \operatorname{sym} \nabla_h \psi_{m+\varepsilon n}^h) \mathrm{d}x \\ &\qquad - \lim_{h \downarrow 0} \frac{1}{2} \int_\Omega \mathbb{A}^h(\operatorname{sym} \imath(m) + \operatorname{sym} \nabla_h \psi_m^h) : (\operatorname{sym} \imath(m) + \operatorname{sym} \nabla_h \psi_m^h) \mathrm{d}x \\ &\quad= \lim_{h \downarrow 0} \frac{1}{2} \int_\Omega \mathbb{A}^h(\operatorname{sym} \imath(m + \varepsilon n) + \operatorname{sym} \nabla_h \psi_{m+\varepsilon n}^h) : \operatorname{sym} \imath(m + \varepsilon n) \mathrm{d}x \\ &\qquad - \lim_{h \downarrow 0} \frac{1}{2} \int_\Omega \mathbb{A}^h(\operatorname{sym} \imath(m) + \operatorname{sym} \nabla_h \psi_m^h) : \operatorname{sym} \imath(m) \mathrm{d}x \\ &\quad= \lim_{h \downarrow 0} \frac{1}{2} \int_\Omega \mathbb{A}^h(\operatorname{sym} \imath(m + \varepsilon n) + \operatorname{sym} \nabla_h \psi_{m+\varepsilon n}^h) : (\operatorname{sym} \imath(m) + \operatorname{sym} \nabla_h \psi_m^h) \mathrm{d}x \\ &\qquad + \lim_{h \downarrow 0} \frac{\varepsilon}{2} \int_\Omega \mathbb{A}^h(\operatorname{sym} \imath(m + \varepsilon n) + \operatorname{sym} \nabla_h \psi_{m+\varepsilon n}^h) : (\operatorname{sym} \imath(n) + \operatorname{sym} \nabla_h \psi_n^h) \mathrm{d}x \\ &\qquad - \lim_{h \downarrow 0} \frac{1}{2} \int_\Omega \mathbb{A}^h(\operatorname{sym} \imath(m) + \operatorname{sym} \nabla_h \psi_m^h) : \operatorname{sym} \imath(m) \mathrm{d}x \\ &\quad= \lim_{h \downarrow 0} \frac{1}{2} \int_\Omega \mathbb{A}^h(\operatorname{sym} \imath(m) + \operatorname{sym} \nabla_h \psi_m^h) : \operatorname{sym} \imath(m + \varepsilon n) \mathrm{d}x \\ &\qquad + \lim_{h \downarrow 0} \frac{\varepsilon}{2} \int_\Omega \mathbb{A}^h(\operatorname{sym} \imath(n) + \operatorname{sym} \nabla_h \psi_n^h) : \operatorname{sym} \imath(m + \varepsilon n) \mathrm{d}x \\ &\qquad - \lim_{h \downarrow 0} \frac{1}{2} \int_\Omega \mathbb{A}^h(\operatorname{sym} \imath(m) + \operatorname{sym} \nabla_h \psi_m^h) : \operatorname{sym} \imath(m) \mathrm{d}x \\ &\quad= \lim_{h \downarrow 0} \varepsilon \int_\Omega \mathbb{A}^h(\operatorname{sym} \imath(m) + \operatorname{sym} \nabla_h \psi_m^h) : \operatorname{sym} \imath(n) \mathrm{d}x \\ &\qquad + \lim_{h \downarrow 0} \frac{\varepsilon^2}{2} \int_\Omega \mathbb{A}^h(\operatorname{sym} \imath(n) + \operatorname{sym} \nabla_h \psi_n^h) : \operatorname{sym} \imath(n) \mathrm{d}x.\end{aligned}$$

Finally, according to the definition (25) and utilizing the uniform L^∞-bound for the sequence of tensors $(\mathbb{A}^h)$, we infer

$$\frac{\delta \mathcal{K}_{(h)}(m)}{\delta m}[n] = \lim_{h\downarrow 0} \int_\Omega \mathbb{A}^h(\operatorname{sym} \imath(m) + \operatorname{sym} \nabla_h \psi_m^h) : \operatorname{sym} \imath(n) \mathrm{d}x\,. \tag{26}$$

3. Derivation of homogenized Euler–Lagrange equations — proof of Theorem 1.1

Taking the L^2-derivative of the energy functional $\mathcal{E}^h$ defined by (1), one finds the Euler–Lagrange equation in the weak form:

$$\frac{\delta \mathcal{E}^h(y^h)}{\delta y^h}[\phi] = \int_\Omega \Big(DW^h(x, \nabla_h y^h) : \nabla_h \phi - h^3(f_2\phi_2 + f_3\phi_3) \Big)\, \mathrm{d}x = 0\,, \tag{27}$$

for all test functions $\phi \in H^1_\omega(\Omega, \mathbb{R}^3)$. Let y^h be a stationary point of $\mathcal{E}^h$, i.e. it satisfies (27). From the frame indifference of W^h it follows that $R^T DW^h(x, RF) = DW^h(x, F)$ for all $R \in \mathrm{SO}(3)$, $F \in \mathbb{R}^{3\times 3}$ and a.e. $x \in \Omega$, which implies (using that $\nabla_h y^h = R^h(I + h^2 G^h)$)

$$DW^h(x, \nabla_h y^h) = R^h DW^h(x, I + h^2 G^h) = h^2 R^h E^h\,. \tag{28}$$

Taylor expansion around the identity gives

$$DW^h(x, I + h^2 G^h) = h^2 D^2 W^h(x, I) G^h + \zeta^h(x, h^2 G^h)\,,$$

where ζ^h is such that $|\zeta^h(\cdot, F)|/|F| \leq r(|F|)$ uniformly in Ω, for all $F \in \mathbb{R}^{3\times 3}$ and $h > 0$. The latter follows from the assumption (11) on admissible composite materials. Since $D^2 W^h(x, I) = \mathbb{A}^h(x)$ and $\mathbb{A}^h(x)$ is a symmetric tensor, the above identity yields

$$E^h(x) = \mathbb{A}^h(x) \operatorname{sym} G^h(x) + \frac{1}{h^2} \zeta^h(x, h^2 G^h)\,, \tag{29}$$

which after employing (19) leads to (recall that $m_d = \Big(u' + \frac{1}{2}((v_2')^2 + (v_3')^2)\Big) e_1 + A' \mathfrak{p}_{x'}$):

$$E^h = \mathbb{A}^h(\operatorname{sym} \imath(m_d) + \operatorname{sym} \nabla_h \psi^h) + \frac{1}{h^2} \zeta^h(\cdot, h^2 G^h) + \mathbb{A}^h o^h\,. \tag{30}$$

3.1. Orthogonality property

In order to identify the fixed part m_d of the symmetrized strain as a stationary point of the limit energy, we first prove the following result.

Lemma 3.1. *Let $(\mathbb{A}^h)$ be a sequence of tensors describing an admissible composite material, let m_d be the fixed part of the symmetrized strain defined by (18), and $(\psi^h) \subset H^1(\Omega, \mathbb{R}^3)$ the corresponding corrector sequence in* (19) *satisfying: $(\psi_1^h, h\psi_2^h, h\psi_3^h) \to 0$ and $\mathfrak{t}(\psi_2^h, \psi_3^h) \to 0$ strongly in the L^2-norm, and $\|\operatorname{sym} \nabla_h \psi^h\|_{L^2(\Omega)} \leq C$. Then, for every sequence $(\varphi^h) \subset H^1(\Omega, \mathbb{R}^3)$ satisfying: $(\varphi_1^h, h\varphi_2^h, h\varphi_3^h) \to 0$ and $\mathfrak{t}(\varphi_2^h, \varphi_3^h) \to 0$ strongly in the L^2-norm, and $(|\operatorname{sym} \nabla_h \varphi^h|^2)$ is equi-integrable, the following* orthogonality property *holds*

$$\lim_{h\downarrow 0} \int_\Omega \mathbb{A}^h(\operatorname{sym} \imath(m_d) + \operatorname{sym} \nabla_h \psi^h) : \operatorname{sym} \nabla_h \varphi^h \mathrm{d}x = 0\,. \tag{31}$$

Proof. Let $(\psi^h) \subset H^1(\Omega,\mathbb{R}^3)$ and $(\varphi^h) \subset H^1(\Omega,\mathbb{R}^3)$ be sequences satisfying the assumptions of the lemma. Applying the Griso's decomposition to the sequence (φ^h) (cf. [22, Corollary 2.3]), there exist sequences $(\Phi^h) \subset H^1((0,L),\mathbb{R}^3_{\rm skw})$, $(\phi^h) \subset H^1(\Omega,\mathbb{R}^3)$ and $(o^h) \subset L^2(\Omega,\mathbb{R}^{3\times 3})$ satisfying:

$$\operatorname{sym}\nabla_h\varphi^h = \operatorname{sym}\imath((\Phi^h)'\mathfrak{p}_{x'}) + \operatorname{sym}\nabla_h\phi^h + o^h\,, \tag{32}$$

$\Phi^h \to 0$, $\phi^h \to 0$, $o^h \to 0$ strongly in the L^2-norm, and

$$\|\Phi^h\|_{H^1(0,L)} + \|\phi^h\|_{L^2(\Omega)} + \|\nabla_h\phi^h\|_{L^2(\Omega)} \le C\|\operatorname{sym}\nabla_h\varphi^h\|_{L^2(\Omega)}\,, \quad \forall h>0\,. \tag{33}$$

Furthermore, there exist subsequences (Φ^h) and (ϕ^h) (still denoted by (h)) and sequences $(\tilde\Phi^h) \subset H^1((0,L),\mathbb{R}^3)$ and $(\tilde\phi^h) \subset H^1(\Omega,\mathbb{R}^3)$ such that $|\{\Phi^h \neq \tilde\Phi^h\} \cup \{(\Phi^h)' \neq (\tilde\Phi^h)'\}| \to 0$ and $|\{\phi^h \neq \tilde\phi^h\} \cup \{\nabla\phi^h \neq \nabla\tilde\phi^h\}| \to 0$ as $h \downarrow 0$, and the sequences $(|(\tilde\Phi^h)'|^2)$ and $(|\nabla_h\tilde\phi^h|^2)$ are equi-integrable (cf. [16] and [22, Lemma 2.17]). Notice that, due to equi-integrability of $(|\operatorname{sym}\nabla_h\varphi^h|^2)$, the decomposition (32) is valid with (Φ^h) and (ϕ^h) replaced by $(\tilde\Phi^h)$ and $(\tilde\phi^h)$. Also, observe that without loss of generality we can assume that for each h, $\tilde\Phi^h$ and $\tilde\phi^h$ are smooth. The rest of the proof will be divided into two parts showing the property (31) using the decomposition (32) with $(\tilde\Phi^h)$ and $(\tilde\phi^h)$.

Part 1. The equi-integrability property of the sequence $(\tilde\phi^h)$ allows us to modify each $\tilde\phi^h$ to zero near the boundary (cf. [32, Lemma 3.6]), thus, making it an eligible test function in the Euler–Lagrange equation (27). Using the identity (30) and the modified $\tilde\phi^h$ as a test function in the Euler–Lagrange equation (27), after division by h^2, we obtain (according to (28))

$$\begin{aligned}\int_\Omega R^h\mathbb{A}^h(\operatorname{sym}\imath(m_d) + \operatorname{sym}\nabla_h\psi^h) : \nabla_h\tilde\phi^h\mathrm{d}x &= \int_\Omega R^h\left(E^h - \frac{1}{h^2}\zeta^h(\cdot,h^2G^h) - \mathbb{A}^h o^h\right) : \nabla_h\tilde\phi^h\mathrm{d}x\\ &= \int_\Omega h(f_2\tilde\phi_2^h + f_3\tilde\phi_3^h) - \int_\Omega R^h\left(\frac{1}{h^2}\zeta^h(\cdot,h^2G^h) + \mathbb{A}^h o^h\right) : \nabla_h\tilde\phi^h\mathrm{d}x\,.\end{aligned}$$

Obviously, the first integral on the right-hand side and the second term in the second integral converge to 0 as $h \downarrow 0$. Let us examine the term

$$\frac{1}{h^2}\int_\Omega R^h\zeta^h(\cdot,h^2G^h) : \nabla_h\tilde\phi^h\mathrm{d}x\,.$$

Define the sets $S_h^\alpha := \{x \in \Omega \ : \ h^2|G^h(x)| \le h^\alpha\}$ for $0<\alpha<2$. On S_h^α we have

$$\frac{|\zeta^h(\cdot,h^2G^h)|}{h^2|G^h|}|G^h| \le \sup\left\{\frac{|\zeta^h(\cdot,h^2\tilde G^h)|}{h^2|\tilde G^h|} \ : \ h^2|\tilde G^h| \le h^\alpha\right\}|G^h| \le r(h^\alpha)|G^h|\,.$$

Therefore,

$$\frac{1}{h^2}\left|\int_{S_h^\alpha} R^h\zeta^h(\cdot,h^2G^h) : \nabla_h\tilde\phi^h\mathrm{d}x\right| \le r(h^\alpha)\|R^h\|_{L^\infty(\Omega)}\|G^h\|_{L^2(\Omega)}\|\nabla_h\tilde\phi^h\|_{L^2(\Omega)} \le Cr(h^\alpha) \to 0\,,$$

as $h \downarrow 0$. On the other hand, on $\Omega\backslash S_h^\alpha$ we have a pointwise a.e. bound

$$\frac{1}{h^2}|\zeta^h(\cdot,h^2G^h)| \le C|G^h| \quad \text{a.e. on } \Omega\backslash S_h^\alpha\,,$$

which in fact holds pointwise a.e. on Ω. This follows by the traingle inequality from (29) using (20) and $|\mathbb{A}^h(x)G^h(x)| \leq \beta|G^h(x)|$ for a.e. $x \in \Omega$. Therefore, using the Cauchy-Schwarz we find

$$\frac{1}{h^2}\left|\int_{\Omega\backslash S_h^\alpha} R^h\zeta^h(\cdot, h^2G^h) : \nabla_h\tilde{\phi}^h \mathrm{d}x\right| \leq C\int_{\Omega\backslash S_h^\alpha} |G^h||\nabla_h\tilde{\phi}^h|\mathrm{d}x$$
$$\leq C\left(\int_{\Omega\backslash S_h^\alpha}|G^h|^2\mathrm{d}x\right)^{1/2}\left(\int_{\Omega\backslash S_h^\alpha}|\nabla_h\tilde{\phi}^h|^2\mathrm{d}x\right)^{1/2} \to 0\,.$$

The latter statement (convergence to zero) follows by the equi-integrability property and the fact that $|\Omega\backslash S_h^\alpha| \to 0$, which follows from the Chebyshev inequality. Thus, we have shown

$$\lim_{h\downarrow 0}\int_\Omega R^h\mathbb{A}^h(\operatorname{sym}\imath(m_d) + \operatorname{sym}\nabla_h\psi^h) : \nabla_h\tilde{\phi}^h\mathrm{d}x = 0\,.$$

Since $R^h \to I$ strongly in the L^∞-norm, it follows that

$$\lim_{h\downarrow 0}\int_\Omega \mathbb{A}^h(\operatorname{sym}\imath(m_d) + \operatorname{sym}\nabla_h\psi^h) : \nabla_h\tilde{\phi}^h\mathrm{d}x = 0\,,$$

while the symmetry property of $\mathbb{A}^h(\operatorname{sym}\imath(m_d) + \operatorname{sym}\nabla_h\psi^h)$ eventually implies

$$\lim_{h\downarrow 0}\int_\Omega \mathbb{A}^h(\operatorname{sym}\imath(m_d) + \operatorname{sym}\nabla_h\psi^h) : \operatorname{sym}\nabla_h\tilde{\phi}^h\mathrm{d}x = 0\,. \tag{34}$$

Part 2. Again, the equi-integrability property of the sequence $(\tilde{\Phi}^h)$ allows us to modify each $\tilde{\Phi}^h$ to zero near the boundary, thus, making the following functions

$$\hat{\phi}^h(x) = \Big(\tilde{\Phi}^h_{12}(x_1)x_2 + \tilde{\Phi}^h_{13}(x_1)x_3\,, -\frac{1}{h}\int_0^{x_1}\tilde{\Phi}^h_{12}(s)\mathrm{d}s + \tilde{\Phi}^h_{23}(x_1)x_3\,, \tag{35}$$
$$-\frac{1}{h}\int_0^{x_1}\tilde{\Phi}^h_{13}(s)\mathrm{d}s - \tilde{\Phi}^h_{23}(x_1)x_2\Big)\,,$$

eligible test functions in the Euler–Lagrange equation (27). One easily calculates

$$\operatorname{sym}\nabla_h\hat{\phi}^h = \begin{pmatrix} (\tilde{\Phi}^h_{12})'(x_1)x_2 + (\tilde{\Phi}^h_{13})'(x_1)x_3 & \frac{1}{2}(\tilde{\Phi}^h_{23})'(x_1)x_3 & -\frac{1}{2}(\tilde{\Phi}^h_{23})'(x_1)x_2 \\ \frac{1}{2}(\tilde{\Phi}^h_{23})'(x_1)x_3 & 0 & 0 \\ -\frac{1}{2}(\tilde{\Phi}^h_{23})'(x_1)x_2 & 0 & 0 \end{pmatrix}$$
$$= \operatorname{sym}\imath((\tilde{\Phi}^h)'\mathfrak{p}_{x'})\,.$$

Using $\hat{\phi}^h$ as a test function in (27) together with the symmetry property of the matrix $DW^h(\cdot, F)F^T$, we obtain

$$\frac{1}{h^2}\int_\Omega DW^h(x, R^h(I + h^2G^h)) : \nabla_h\hat{\phi}^h\mathrm{d}x$$
$$= \frac{1}{h^2}\int_\Omega R^hDW^h(x, I + h^2G^h)(I + h^2G^h)^T(R^h)^T : \operatorname{sym}\nabla_h\hat{\phi}^h\mathrm{d}x$$
$$-\frac{1}{h^2}\int_\Omega R^hDW^h(x, I + h^2G^h)\left((R^h)^T - I + h^2(G^h)^T(R^h)^T\right) : \nabla_h\hat{\phi}^h\mathrm{d}x$$
$$= \int_\Omega R^hE^h(I + h^2G^h)^T(R^h)^T : \operatorname{sym}\imath((\tilde{\Phi}^h)'\mathfrak{p}_{x'})\mathrm{d}x$$
$$-\int_\Omega R^hE^h\left(\frac{1}{h}((R^h)^T - I) + h(G^h)^T(R^h)^T\right) : h\nabla_h\hat{\phi}^h\mathrm{d}x\,.$$

Therefore, the Euler–Lagrange equation becomes

$$\int_\Omega R^h E^h (I + h^2 G^h)^T (R^h)^T : \operatorname{sym} \imath((\tilde{\Phi}^h)' \mathfrak{p}_{x'}) \mathrm{d}x$$
$$= \int_\Omega R^h E^h \left(\frac{1}{h}((R^h)^T - I) + h(G^h)^T (R^h)^T \right) : h\nabla_h \hat{\phi}^h \mathrm{d}x + h \int_\Omega (f_2 \hat{\phi}_2^h + f_3 \hat{\phi}_3^h) \mathrm{d}x \,. \tag{36}$$

Since $(h\hat{\phi}_2^h, h\hat{\phi}_3^h) \to 0$ strongly in the L^2-norm, the force term vanishes at the limit. According to (33), $\|(\tilde{\Phi}^h)'\|_{L^2(0,L)}$ is uniformly bounded implying the strong convergence $h\nabla_h \hat{\phi}^h \to 0$ in the L^2-norm, therefore,

$$\lim_{h \downarrow 0} \frac{1}{h} \int_\Omega R^h E^h ((R^h)^T - I) : h\nabla_h \hat{\phi}^h \mathrm{d}x = 0 \,.$$

In order to infer zero at the limit $h \downarrow 0$ for the remaining term on the right-hand side in (36), namely

$$h \int_\Omega R^h E^h (G^h)^T (R^h)^T : h\nabla_h \hat{\phi}^h \mathrm{d}x \,,$$

we need to replace the sequence $(\tilde{\Phi}^h)$ with the one obtained by means of Lemma A.1. We take the sequence $s_h = 1/\sqrt{h}$ and obtain a sequence $(\tilde{\tilde{\Phi}}^h)$ satisfying $\|\tilde{\tilde{\Phi}}^h\|_{W^{1,\infty}(0,L)} \le C s_h$ for some $C > 0$. Notice that we have $\|(\tilde{\tilde{\Phi}}^h)' - (\tilde{\Phi}^h)'\|_{L^2} \to 0$. We easily conclude $\|h\nabla_h \tilde{\hat{\phi}}^h\|_{L^\infty} \le C s_h$ where, in view of (35), notation $\tilde{\hat{\phi}}^h$ is self-explaining. From the latter we conclude that

$$\lim_{h \downarrow 0} h \int_\Omega R^h E^h (G^h)^T (R^h)^T : h\nabla_h \tilde{\hat{\phi}}^h \mathrm{d}x = 0 \,,$$

which implies

$$\lim_{h \downarrow 0} \int_\Omega R^h E^h (I + h^2 G^h)^T (R^h)^T : \operatorname{sym} \imath((\tilde{\tilde{\Phi}}^h)' \mathfrak{p}_{x'}) \mathrm{d}x = 0 \,.$$

Obviously,

$$\lim_{h \downarrow 0} \int_\Omega h^2 R^h E^h (G^h)^T (R^h)^T : \operatorname{sym} \imath((\tilde{\tilde{\Phi}}^h)' \mathfrak{p}_{x'}) \mathrm{d}x = 0 \,,$$

and therefore,

$$\lim_{h \downarrow 0} \int_\Omega R^h E^h (R^h)^T : \operatorname{sym} \imath((\tilde{\tilde{\Phi}}^h)' \mathfrak{p}_{x'}) \mathrm{d}x = 0 \,. \tag{37}$$

Next, we prove that

$$\lim_{h \downarrow 0} \int_\Omega E^h : \operatorname{sym} \imath((\tilde{\tilde{\Phi}}^h)' \mathfrak{p}_{x'}) \mathrm{d}x = 0 \,. \tag{38}$$

This follows by writing

$$\int_\Omega E^h : \operatorname{sym} \imath((\tilde{\tilde{\Phi}}^h)' \mathfrak{p}_{x'}) \mathrm{d}x = \int_\Omega \left(R^h + (I - R^h) \right) E^h \left(R^h + (I - R^h) \right)^T : \operatorname{sym} \imath((\tilde{\tilde{\Phi}}^h)' \mathfrak{p}_{x'}) \mathrm{d}x \,,$$

and using the convergence result (37) with the fact that $R^h \to I$ strongly in the L^∞-norm. Now, recall that

$$\mathbb{A}^h (\operatorname{sym} \imath(m_d) + \operatorname{sym} \nabla_h \psi^h) = E^h - \frac{1}{h^2} \zeta^h(\cdot, h^2 G^h) + o^h \,,$$

where $o^h \to 0$ strongly in the L^2-norm. Using truncation arguments on the sets S_h^α and its complement, as in the first part of the proof, we conclude

$$\lim_{h\downarrow 0}\int_\Omega \frac{1}{h^2}\zeta^h(\cdot, h^2G^h) : \operatorname{sym}\imath((\tilde{\tilde{\Phi}}^h)'\mathfrak{p}_{x'})\mathrm{d}x = 0\,.$$

Since $\lim_{h\downarrow 0}\int_\Omega o^h : \operatorname{sym}\imath((\tilde{\Phi}^h)'\mathfrak{p}_{x'})\mathrm{d}x = 0$, convergence result (38) implies

$$\lim_{h\downarrow 0}\int_\Omega \mathbb{A}^h(\operatorname{sym}\imath(m_d) + \operatorname{sym}\nabla_h\psi^h) : \operatorname{sym}\imath((\tilde{\tilde{\Phi}}^h)'\mathfrak{p}_{x'})\mathrm{d}x = 0.$$

From this it follows

$$\lim_{h\downarrow 0}\int_\Omega \mathbb{A}^h(\operatorname{sym}\imath(m_d) + \operatorname{sym}\nabla_h\psi^h) : \operatorname{sym}\imath((\tilde{\Phi}^h)'\mathfrak{p}_{x'})\mathrm{d}x = 0\,. \qquad \square$$

3.2. Identification of the limit Euler–Lagrange equations

Let us now more precisely identify terms in the Euler–Lagrange equation (27) and consider the limit when $h\downarrow 0$. The same reasoning as in Part 2 of the proof of Lemma 3.1 gives, after division by h^2, the Euler–Lagrange equation (27) in the form

$$\begin{aligned}&\int_\Omega R^hE^h(I+h^2G^h)^T(R^h)^T : \operatorname{sym}\nabla_h\phi^h\,\mathrm{d}x\\ &\qquad = \int_\Omega R^hE^h\left(\frac{1}{h}((R^h)^T - I) + h(G^h)^T(R^h)^T\right) : h\nabla_h\phi^h\,\mathrm{d}x + h\int_\Omega (f_2\phi_2^h + f_3\phi_3^h)\mathrm{d}x\,,\end{aligned} \tag{39}$$

for all test functions $\phi^h \in H^1_\omega(\Omega,\mathbb{R}^3)$. The aim is now to identify the limit equation in (39) as $h\downarrow 0$. We will do the computations under the assumption that $\limsup_{h\downarrow 0}\|\operatorname{sym}\nabla_h\phi^h\|_{L^\infty} < \infty$. Using the facts that, up to a term converging to zero strongly in the L^2-norm,

$$E^h = \mathbb{A}^h(\operatorname{sym}\imath(m_d) + \operatorname{sym}\nabla_h\psi^h) + \frac{1}{h^2}\zeta^h(\cdot, h^2G^h)\,, \tag{40}$$

$R^h \to I$ strongly in the L^∞-norm, and

$$\lim_{h\downarrow 0}\int_\Omega h^2R^hE^h(G^h)^T(R^h)^T : \operatorname{sym}\nabla_h\phi^h\,\mathrm{d}x = 0\,,$$

the limit $h\downarrow 0$ (if it exists) of

$$\int_\Omega R^hE^h(I+h^2G^h)^T(R^h)^T : \operatorname{sym}\nabla_h\phi^h\,\mathrm{d}x$$

equals the limit

$$\lim_{h\downarrow 0}\int_\Omega \big(\mathbb{A}^h(\operatorname{sym}\imath(m_d) + \operatorname{sym}\nabla_h\psi^h) + \frac{1}{h^2}\zeta^h(\cdot, h^2G^h)\big) : \operatorname{sym}\nabla_h\phi^h\,\mathrm{d}x\,.$$

The remainder term $\frac{1}{h^2}\int_\Omega \zeta^h(\cdot, h^2G^h) : \operatorname{sym}\nabla_h\phi^h\,\mathrm{d}x$ vanishes in the same way as in the proof of Lemma 3.1, and in the limit as $h\downarrow 0$, equation (39) reduces to

$$\lim_{h\downarrow 0}\int_\Omega \mathbb{A}^h(\operatorname{sym}\imath(m_d) + \operatorname{sym}\nabla_h\psi^h) : \operatorname{sym}\nabla_h\phi^h\,\mathrm{d}x$$
$$= \lim_{h\downarrow 0}\left(\int_\Omega R^hE^h\left(\frac{1}{h}((R^h)^T - I) + h(G^h)^T(R^h)^T\right) : h\nabla_h\phi^h\,\mathrm{d}x + h\int_\Omega (f_2\phi_2^h + f_3\phi_3^h)\mathrm{d}x\right). \tag{41}$$

First, consider the test function $\phi(x) = \int_0^{x_1}\phi_{11}(t)e_1$ with ϕ_{11} smooth. Since $\phi_2 = \phi_3 = 0$, $\operatorname{sym}\nabla_h\phi = \phi_{11}(x_1)e_1\otimes e_1$, and $h\nabla_h\phi \to 0$ strongly in the L^2-norm, (41) amounts to

$$\lim_{h\downarrow 0}\int_\Omega \mathbb{A}^h(\operatorname{sym}\imath(m_d) + \operatorname{sym}\nabla_h\psi^h) : \phi_{11}(x_1)e_1\otimes e_1\,\mathrm{d}x = 0\,. \tag{42}$$

Next, consider test functions of the form $\phi_{ij}^h(x) = hx_j\phi_{ij}(x_1)e_i$ for $i = 1,2,3$, $j = 2,3$, where ϕ_{ij} is smooth with $\phi_{ij}(0) = 0$. The functions ϕ_{ij}^h obviously satisfy $(\phi_{ij,1}^h, h\phi_{ij,2}^h, h\phi_{ij,3}^h) \to 0$ and $\mathfrak{t}(\phi_{ij,2}^h, \phi_{ij,3}^h) \to 0$ strongly in the L^2-norm. Calculating

$$\operatorname{sym}\nabla_h\phi_{ij}^h = \operatorname{sym}\left(hx_j\phi_{ij}'e_i \,|\, \delta_{2j}\phi_{ij}e_i \,|\, \delta_{3j}\phi_{ij}e_i\right)\,,$$

we easily conclude from (41) that

$$\lim_{h\downarrow 0}\int_\Omega \mathbb{A}^h(\operatorname{sym}\imath(m_d) + \operatorname{sym}\nabla_h\psi^h) : \phi_{ij}(x_1)e_i\otimes e_j\,\mathrm{d}x = 0\,, \tag{43}$$

for all $i = 1,2,3$, $j = 2,3$. Finally, consider the test function given by

$$\phi^h(x) = \left(\Phi_{12}(x_1)x_2 + \Phi_{13}(x_1)x_3\,, \frac{1}{h}\int_0^{x_1}\Phi_{21}(s)\mathrm{d}s + \Phi_{23}(x_1)x_3\,, \frac{1}{h}\int_0^{x_1}\Phi_{31}(s)\mathrm{d}s + \Phi_{32}(x_1)x_2\right),$$

where $\Phi : [0, L] \to \mathbb{R}^{3\times 3}_{\mathrm{skw}}$ is smooth and $\Phi(0) = 0$. On the right-hand side of (41), using the convergence results: $R^h \to I$ strongly in the L^∞-norm, $hG^h \to 0$ strongly in the L^2-norm, $A^h \to A$ (the definition of A is given in (16)) strongly in the L^∞-norm, as well as the approximation identity (40) for E^h, we are left with

$$\lim_{h\downarrow 0}\int_\Omega \mathbb{A}^h(\operatorname{sym}\imath(m_d) + \operatorname{sym}\nabla_h\psi^h)A^T : \Phi\,\mathrm{d}x$$
$$+ \int_0^L\left(f_2(x_1)\int_0^{x_1}\Phi_{21}(s)\mathrm{d}s + f_3(x_1)\int_0^{x_1}\Phi_{31}(s)\mathrm{d}s\right)\mathrm{d}x_1\,.$$

Let us now consider the first term of the obtained expression. Due to the real matrix identity $XY : Z = -X : ZY$, for Y being skew-symmetric matrix, the first term equals (up to a minus sign)

$$\lim_{h\downarrow 0}\int_\Omega \mathbb{A}^h(\operatorname{sym}\imath(m_d) + \operatorname{sym}\nabla_h\psi^h) : \Phi A\,\mathrm{d}x\,,$$

and since the first matrix is symmetric, the latter in fact equals to

$$\lim_{h\downarrow 0}\int_\Omega \mathbb{A}^h(\operatorname{sym}\imath(m_d) + \operatorname{sym}\nabla_h\psi^h) : \operatorname{sym}(\Phi A)\mathrm{d}x\,. \tag{44}$$

The matrix ΦA can be explicitly computed, and its symmetric part is given by

$$\operatorname{sym}(\Phi A) = \begin{pmatrix} \Phi_{12}v_2' + \Phi_{13}v_3' & \frac{1}{2}(\Phi_{23}v_3' + \Phi_{13}w) & -\frac{1}{2}(\Phi_{23}v_2' + \Phi_{12}w) \\ \frac{1}{2}(\Phi_{23}v_3' + \Phi_{13}w) & \Phi_{12}v_2' + \Phi_{23}w & \frac{1}{2}(\Phi_{13}v_2' + \Phi_{12}v_3') \\ -\frac{1}{2}(\Phi_{23}v_2' + \Phi_{12}w) & \frac{1}{2}(\Phi_{13}v_2' + \Phi_{12}v_3') & \Phi_{13}v_3' + \Phi_{23}w \end{pmatrix}.$$

Defining the sequence of test functions (φ_A^h) by

$$\varphi_A^h(x) = hx_2 \begin{pmatrix} \Phi_{23}v_3' + \Phi_{13}w \\ \Phi_{12}v_2' + \Phi_{23}w \\ \Phi_{13}v_2' + \Phi_{12}v_3' \end{pmatrix} + hx_3 \begin{pmatrix} -\Phi_{23}v_2' - \Phi_{12}w \\ \Phi_{13}v_2' + \Phi_{12}v_3' \\ \Phi_{13}v_3' + \Phi_{23}w \end{pmatrix},$$

it is straightforward to check that

$$\operatorname{sym}(\Phi A) = \operatorname{sym}\nabla_h\varphi_A^h + (\Phi_{12}v_2' + \Phi_{13}v_3')e_1 \otimes e_1 + o^h, \tag{45}$$

where o^h converges to zero strongly in the L^2-norm as $h \downarrow 0$. Observe that the sequence of test functions (φ_A^h) satisfies $(\varphi_{A,1}^h, h\varphi_{A,2}^h, h\varphi_{A,3}^h) \to 0$ and $\mathfrak{t}(\varphi_{A,2}^h, \varphi_{A,3}^h) \to 0$ strongly in the L^2-norm. Utilizing (45) in expression (44), we confer that due to the orthogonality property (31), convergence result (42) and strongly to zero convergence of o^h, the term in (44) vanishes in the limit as $h \downarrow 0$. Since,

$$\operatorname{sym}\nabla_h\phi^h = \begin{pmatrix} \Phi_{12}'(x_1)x_2 + \Phi_{13}'(x_1)x_3 & \frac{1}{2}\Phi_{23}'(x_1)x_3 & -\frac{1}{2}\Phi_{23}'(x_1)x_2 \\ \frac{1}{2}\Phi_{23}'(x_1)x_3 & 0 & 0 \\ -\frac{1}{2}\Phi_{23}'(x_1)x_2 & 0 & 0 \end{pmatrix} = \operatorname{sym}\imath(\Phi'\mathfrak{p}_{x'}), \tag{46}$$

the left-hand side in (41) can be written as

$$\lim_{h\downarrow 0}\int_\Omega \mathbb{A}^h(\operatorname{sym}\imath(m_d) + \operatorname{sym}\nabla_h\psi^h) : \operatorname{sym}\imath(\Phi'\mathfrak{p}_{x'})\,\mathrm{d}x. \tag{47}$$

Combining (42), (43) and (47), the resolved limiting Euler–Lagrange equation (41) reads

$$\begin{aligned}\lim_{h\downarrow 0}\int_\Omega \mathbb{A}^h(\operatorname{sym}\imath(m_d) + \operatorname{sym}\nabla_h\psi^h) : \operatorname{sym}\Big(\phi_{11}e_1\otimes e_1 + \sum_{i=1,j=2}^{3}\phi_{ij}e_i\otimes e_j + \imath(\Phi'\mathfrak{p}_{x'})\Big)\,\mathrm{d}x \\ = -\int_0^L (f_2\tilde\Phi_{12} + f_3\tilde\Phi_{13})\mathrm{d}x_1,\end{aligned} \tag{48}$$

where $\tilde\Phi_{1j}(x_1) = \int_0^{x_1}\Phi_{1j}(s)\mathrm{d}s$ for $j = 2,3$. Now, to conclude the proof we claim, that the obtained equation (neglecting the terms $\sum_{i=1,j=2}^3 \phi_{ij}e_i\otimes e_j$ in the first sum due to (43)) can be interpreted as

$$\frac{\delta\mathcal{K}_{(h)}}{\delta m}(m_d)\Big[\phi_{11}e_1 + \Phi'\mathfrak{p}_{x'}\Big] = -\int_0^L (f_2\tilde\Phi_{12} + f_3\tilde\Phi_{13})\mathrm{d}x_1. \tag{49}$$

Notice that, due to (42), we could also neglect the first term in (48), but we will make use of it later. Since $(\operatorname{sym}\nabla_h\psi^h)$ is bounded in the L^2-norm, according to [22, Lemma 2.17], there exists a subsequence (still denoted by (h)) and sequence $(\tilde\psi^h)$ such that $(|\operatorname{sym}\nabla_h\tilde\psi^h|^2)$ is equi-integrable and $\|\operatorname{sym}\nabla_h\psi^h - \operatorname{sym}\nabla_h\tilde\psi^h\|_{L^2(O^h)} \to 0$, where $O^h \subset \Omega$ such that $|\Omega\backslash O^h| \to 0$.

From (48) we see that the same limit equation will be obtained if we replace the corrector sequence (ψ^h) by $(\tilde{\psi}^h)$. Let $(\psi^h_{m_d})$ be the relaxation sequence for m_d from Lemma 2.5. Using the coercivity of Q^h and the orthogonality properties (23) and (31) of both sequences $(\psi^h_{m_d})$ and $(\tilde{\psi}^h)$, respectively, we find that

$$\begin{aligned}\alpha\|\operatorname{sym}\nabla_h(\psi^h_{m_d}-\tilde{\psi}^h)\|^2_{L^2} &\leq \int_\Omega Q^h(x,\operatorname{sym}\nabla_h(\psi^h_{m_d}-\tilde{\psi}^h))\mathrm{d}x\\ &= \frac{1}{2}\int_\Omega \mathbb{A}^h(\operatorname{sym}\imath(m_d)+\operatorname{sym}\nabla_h\psi^h_{m_d}) : \operatorname{sym}\nabla_h(\psi^h_{m_d}-\tilde{\psi}^h)\mathrm{d}x\\ &\quad -\frac{1}{2}\int_\Omega \mathbb{A}^h(\operatorname{sym}\imath(m_d)+\operatorname{sym}\nabla_h\psi^h) : \operatorname{sym}\nabla_h(\psi^h_{m_d}-\tilde{\psi}^h)\mathrm{d}x \to 0\end{aligned}$$

as $h\downarrow 0$. Therefore, we can also replace the sequence $(\tilde{\psi}^h)$ by $(\psi^h_{m_d})$ in (48) and (49) follows from (26). To prove the stationarity of the point (u,v_2,v_3,w) for the functional $\mathcal{E}^0$ from the equation (49) we note the following:

$$\begin{aligned}\frac{\delta(\mathcal{K}_{(h)}\circ m_d)}{\delta u}(u,v_2,v_3,w)[u^d] &= \frac{\delta\mathcal{K}_{(h)}}{\delta m}(m_d)[u^d e_1]\,, \quad \forall u^d\in H^1_0(0,L)\,,\\ \frac{\delta(\mathcal{K}_{(h)}\circ m_d)}{\delta v_i}(u,v_2,v_3,w)[v_i^d] &= \frac{\delta\mathcal{K}_{(h)}}{\delta m}(m_d)[(v_i^d)'e_1+\Phi'_{v_i^d}\mathfrak{p}_{x'}]\,,\ \text{for } i=2,3, \quad \forall v_i^d\in H^2_0(0,L)\,,\\ \frac{\delta(\mathcal{K}_{(h)}\circ m_d)}{\delta w}(u,v_2,v_3,w)[w^d] &= \frac{\delta\mathcal{K}_{(h)}}{\delta m}(m_d)[\Phi'_{w^d}\mathfrak{p}_{x'}]\,, \quad \forall w^d\in H^1_0(0,L)\,.\end{aligned}$$

Here

$$\Phi_{v_2^d}=\begin{pmatrix}0 & -(v_2^d)' & 0\\ (v_2^d)' & 0 & 0\\ 0 & 0 & 0\end{pmatrix},\ \Phi_{v_3^d}=\begin{pmatrix}0 & 0 & -(v_3^d)'\\ 0 & 0 & 0\\ (v_3^d)' & 0 & 0\end{pmatrix},\ \Phi_{w^d}=\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & -w^d\\ 0 & w^d & 0\end{pmatrix}.$$

This finishes the proof of Theorem 1.1.

In the subsequent part of the section we identify the limit Euler–Lagrange equations. Recalling the approximation identity (30), the weak convergence $E^h\rightharpoonup E$ in $L^2(\Omega,\mathbb{R}^{3\times3})$, and utilizing convergence properties for the remainder terms, we can pass to the limit in equation (48) and obtain

$$\int_\Omega E:\operatorname{sym}\Big(\phi'_{11}e_1\otimes e_1+\sum_{i=1,j=2}^{3}\phi_{ij}e_i\otimes e_j+\imath(\Phi'\mathfrak{p}_{x'})\Big)\,\mathrm{d}x=-\int_0^L(f_2\tilde{\Phi}_{12}+f_3\tilde{\Phi}_{13})\mathrm{d}x_1\,.$$

In view of identity (46), the latter equals

$$\begin{aligned}\int_\Omega\Big(E_{11}\phi'_{11}+\sum_{i=1,j=2}^{3}E_{ij}\phi_{ij}\ &+x_2E_{11}\Phi'_{12}(x_1)+x_3E_{11}\Phi'_{13}(x_1)\\ &+x_3E_{12}\Phi'_{23}(x_1)-x_2E_{13}\Phi'_{23}(x_1)\Big)\mathrm{d}x=-\int_0^L(f_2\tilde{\Phi}_{12}+f_3\tilde{\Phi}_{13})\mathrm{d}x_1\,.\end{aligned} \tag{50}$$

Using the moment notation (9)–(10) and the fact that $\tilde{\Phi}'_{1j} = \Phi_{1j}$ for $j = 2,3$, (51) becomes

$$\int_0^L \Big(\overline{E}_{11}\phi'_{11} + \sum_{i=1,j=2}^{3} \overline{E}_{ij}\phi_{ij} + \widetilde{E}_{11}\tilde{\Phi}''_{12} + \widehat{E}_{11}\tilde{\Phi}''_{13} + \widehat{E}_{12}\Phi'_{23} - \widetilde{E}_{13}\Phi'_{23}\Big)\mathrm{d}x_1$$

$$= -\int_0^L (f_2\tilde{\Phi}_{12} + f_3\tilde{\Phi}_{13})\mathrm{d}x_1 \,. \qquad (51)$$

Now by the arbitrariness of test functions, we easily derive the corresponding strong formulation for the moments. The zeroth-order moments satisfy

$$\overline{E} = 0 \quad \text{in } (0,L)\,. \qquad (52)$$

The first-order moments $\widetilde{E}_{11}$ and $\widehat{E}_{11}$ satisfy second-order boundary-value problems:

$$\begin{aligned} &\widetilde{E}''_{11} + f_2 = 0 \quad \text{in } (0,L)\,,\\ &\widetilde{E}_{11}(L) = \widetilde{E}'_{11}(L) = 0\,, \end{aligned}$$

and

$$\begin{aligned} &\widehat{E}''_{11} + f_3 = 0 \quad \text{in } (0,L)\,,\\ &\widehat{E}_{11}(L) = \widehat{E}'_{11}(L) = 0\,, \end{aligned}$$

respectively. Finally, the first-order moments $\widehat{E}_{12}$ and $\widetilde{E}_{13}$ satisfy the first-order problem

$$\begin{aligned} &\widehat{E}'_{12} - \widetilde{E}'_{13} = 0 \quad \text{in } (0,L)\,,\\ &\widehat{E}_{12}(L) = \widetilde{E}_{13}(L)\,. \end{aligned}$$

It remains to derive constitutive equations, which connect the moments of the limit stress with limit displacements and twist functions. For $\varrho \in L^2(0,L)$ and $\Psi \in L^2((0,L),\mathbb{R}^{3\times3}_{\mathrm{skw}})$, recall the functional

$$\mathcal{K}_{(h)}(m(\varrho,\Psi)) = \int_0^L Q^0(x_1,\varrho(x_1),\operatorname{axl}\Psi(x_1))\mathrm{d}x_1\,,$$

where $m(\varrho,\Psi)(x) = \varrho(x_1)e_1 + \Psi(x_1)\mathfrak{p}_{x'}$, and the functional

$$\begin{aligned} \mathcal{K}^0_{(h)}(\Psi) &= \int_0^L Q^0_1(x_1,\operatorname{axl}\Psi(x_1))\mathrm{d}x_1 = \int_0^L Q^0(x_1,\varrho_0(x_1,\operatorname{axl}\Psi(x_1)),\operatorname{axl}\Psi(x_1))\mathrm{d}x_1\\ &= \mathcal{K}_{(h)}(m_0(\varrho_0,\Psi))\,, \end{aligned}$$

where $\varrho_0 : (0,L)\times\mathbb{R}^3 \to \mathbb{R}$ is optimal for a given $\operatorname{axl}\Psi$. By Lemma 2.5 (identity (22)), there exist sequences $(\psi^h_m) \subset H^1(\Omega,\mathbb{R}^3)$ and $(\psi^h_0) \subset H^1(\Omega,\mathbb{R}^3)$ such that:

$$\begin{aligned} \mathcal{K}_{(h)}(m(\varrho,\Psi)) &= \lim_{h\downarrow 0}\int_\Omega Q^h(x,\operatorname{sym}\imath(m(\varrho,\Psi)) + \operatorname{sym}\nabla_h\psi^h_m)\mathrm{d}x\,,\\ \mathcal{K}^0_{(h)}(\Psi) &= \lim_{h\downarrow 0}\int_\Omega Q^h(x,\operatorname{sym}\imath(m_0(\varrho,\Psi)) + \operatorname{sym}\nabla_h\psi^h_0)\mathrm{d}x\,. \end{aligned}$$

Using the orthogonality property (23) and tricks as in Section 2.6, we calculate:

$$\frac{\delta \mathcal{K}_{(h)}(m(\varrho,\Psi))}{\delta \varrho}[\phi] = \lim_{h\downarrow 0}\int_\Omega \mathbb{A}^h(\operatorname{sym}\imath(m(\varrho,\Psi)) + \operatorname{sym}\nabla_h \psi_m^h) : \imath(\phi e_1)\mathrm{d}x\,, \tag{53}$$

$$\frac{\delta \mathcal{K}^0_{(h)}(\Psi)}{\delta \Psi}[\Phi] = \lim_{h\downarrow 0}\int_\Omega \mathbb{A}^h(\operatorname{sym}\imath(m_0(\varrho,\Psi)) + \operatorname{sym}\nabla_h \psi_0^h) : \operatorname{sym}\imath(\Phi\mathfrak{p}_{x'})\mathrm{d}x\,, \tag{54}$$

for all $\phi \in C_0^\infty([0,L])$ and $\Phi \in C_0^\infty([0,L],\mathbb{R}^{3\times 3}_{\mathrm{skw}})$. On the other hand, from the representation of the function Q_1^0 as a pointwise quadratic form, we have

$$\frac{\delta \mathcal{K}^0_{(h)}(\Psi)}{\delta \Psi}[\Phi] = \int_0^L \mathbb{A}_1^0(x_1)\operatorname{axl}\Psi(x_1)\cdot \operatorname{axl}\Phi(x_1)\mathrm{d}x_1\,. \tag{55}$$

Now, if we consider $m_d(x) = (u' + \frac{1}{2}((v_2')^2 + (v_3')^2))e_1 + A'\mathfrak{p}_{x'}$, it follows from formulae (42) and (53) that

$$\frac{\delta \mathcal{K}_{(h)}(m(a,A'))}{\delta \varrho}[\phi] = 0$$

for all $\phi \in C_0^\infty([0,L])$, where $a(x_1) = u' + \frac{1}{2}((v_2')^2 + (v_3')^2)$. In particular, this implies the optimality of the function a for matrix function A' in the sense that $Q_1^0(\cdot,\operatorname{axl}A') = Q^0(\cdot,a,\operatorname{axl}A')$. Equating expressions in (54) and (55) for $\Psi = A'$ and $\varrho_0 = a$, we obtain the identity

$$\int_0^L \mathbb{A}_1^0(x_1)\operatorname{axl}A'(x_1)\cdot\operatorname{axl}\Phi(x_1)\mathrm{d}x_1 = \int_\Omega E : \imath(\Phi\mathfrak{p}_{x'})\mathrm{d}x\,,$$

for all $\Phi \in C_0^\infty([0,L],\mathbb{R}^{3\times 3}_{\mathrm{skw}})$. From the latter we recognize the following system

$$\begin{aligned} -(\mathbb{A}_1^0\operatorname{axl}A')_3 &= \widetilde{E}_{11}\,,\\ (\mathbb{A}_1^0\operatorname{axl}A')_2 &= \widehat{E}_{11}\,,\\ -(\mathbb{A}_1^0\operatorname{axl}A')_1 &= \widehat{E}_{12} - \widetilde{E}_{13}\,, \end{aligned}$$

which is a linear second-order system for the limit displacements v_2, v_3 and the limit twist function w, and which needs to be accompanied by the following boundary conditions $v_i(0) = v_i'(L) = 0$ for $i = 2,3$, and $w(0) = 0$. The obtained boundary-value problem represents the homogenized Euler–Lagrange equations for the von Kármán rod model. Finally, the scaled displacement u can deduced from the optimality property of the function a for the matrix function A' and the initial condition $u(0) = 0$.

4. Stochastic Homogenization

In this section we will give an explicit cell formula for the quadratic form Q^0 (limit energy density in expression (24)) under the assumption of random material along the characteristic dimension of the rod. Providing the cell formula for the limit energy in the stochastic setting, we will also recover periodic and almost periodic structures. The methods we are using here are largely based on works [12], [20] and [33]. Firstly, we will introduce the general notion and tools of stochastic homogenization, thereafter we will explore the tools needed for thin structures and finally derive and prove the cell formulae.

4.1. Stochastic homogenization

Definition 4.1. *A family $(T_x)_{x\in\mathbb{R}^n}$ of measurable bijective mappings $T_x\colon \Xi \to \Xi$ on the probability space $(\Xi, \mathscr{F}, \mathbb{P})$ is called a* dynamical system *on Ξ with respect to $\mathbb{P}$ if:*

1. *T is additive, i.e. $T_x \circ T_y = T_{x+y}$ for all $x, y \in \mathbb{R}^n$;*
2. *T is measure- and measurability-preserving, i.e. $T_x B$ is measurable and $\mathbb{P}(T_x B) = \mathbb{P}(B)$ for all $x \in \mathbb{R}^n$ and $B \in \mathscr{F}$;*
3. *The mapping $\mathcal{A}\colon \Xi \times \mathbb{R}^n \to \Xi$, defined by $\mathcal{A}(\rho, x) = T_x(\rho)$, is measurable in the pair of σ-algebras $(\mathscr{F} \times \mathcal{L}^n, \mathscr{F})$, where $\mathcal{L}^n$ denotes the family of Lebesgue measurable sets.*

The key property, which will allow us to derive the cell formula, is ergodicity.

Definition 4.2. *A dynamical system T is called* ergodic*, if one of the following (equivalent) conditions is fulfilled:*

1. *If $f\colon \Xi \to \Xi$ is measurable s.t. $f(\rho) = f(T_x\rho)$ for all $x \in \mathbb{R}^n$ and a.e. $\rho \in \Xi$, then f is $\mathbb{P}$-a.e. equal to a constant.*
2. *If for some $B \in \mathscr{F}$ for all $x \in \mathbb{R}^n$ the set $(T_x B \cup B) \setminus (T_x B \cap B)$ is a null set, then $\mathbb{P}(B) \in \{0, 1\}$.*

One of the most important consequences of ergodicity is the famous Birkhoff's ergodicity theorem:

Theorem 4.1. *Let T be an ergodic, dynamical system and $g \in L^1(\Xi)$. Then*

$$\lim_{t\to\infty} \frac{1}{t^n|A|} \int_{tA} g(T_x\widetilde{\rho})\mathrm{d}x = \int_\Xi g(\rho)\mathrm{d}\mathbb{P}(\rho) \tag{56}$$

for almost all $\widetilde{\rho}$, for all bounded Borel sets $A \subset \mathbb{R}^n$ with $|A| > 0$.

Let $L^p(\Xi)$ denotes the set of measurable p-integrable functions $b\colon \Xi \to \mathbb{R}$. In order to guarantee that the spaces $L^p(\Xi)$ for $p \geq 1$ are separable we assume that the σ-algebra $\mathscr{F}$ is countably generated. The dynamical system allows for more structure on the space Ξ. Denote by $U(x)$ the unitary operator

$$U(x) : L^2(\Xi) \to L^2(\Xi), \quad U(x)b = b \circ T_x\,.$$

If for $b \in L^2(\Xi)$ and $1 \leq k \leq n$ the limit

$$\lim_{h\downarrow 0} \frac{b(T_{h\cdot e_k}\rho) - b(\rho)}{h}$$

exists in the L^2-sense, then we call it the k-th derivative of b and denote it by $D_k b$. The operators D_k are infinitesimal generators of the maps T_{x_k}. Thus, $iD_1, \ldots, iD_n$ are commuting, self-adjoint, closed and densely defined linear operators on the separable Hilbert space $L^2(\Xi)$. Let $\mathcal{D}_k(\Xi)$ denote the domain of the operator D_k, and define the space $W^{1,2}(\Xi)$ as

$$W^{1,2}(\Xi) := \mathcal{D}_1(\Xi) \cap \ldots \cap \mathcal{D}_n(\Xi),$$

equipped with norm

$$\|b\|^2_{W^{1,2}(\Xi)} = \|b\|^2_{L^2(\Xi)} + \sum_{k=1}^{n} \|D_i b\|^2_{L^2(\Xi)}.$$

We also define the semi-norm

$$|b|^2_{W^{1,2}(\Xi)} = \sum_{k=1}^{n} \|D_i b\|^2_{L^2(\Xi)}\,,$$

and analogously the following Sobolev-type spaces:

$$\begin{aligned} W^{k,2}(\Xi) &:= \{b \in L^2(\Xi) \,:\, D_1^{\alpha_1} \dots D_n^{\alpha_n} b \in L^2(\Xi),\ \alpha_1 + \dots + \alpha_n \le k\}\,, \\ W^{\infty,2}(\Xi) &:= \bigcap_{k \ge 0} W^{k,2}(\Xi)\,. \end{aligned}$$

Furthermore, we define the set of *stochastically smooth* functions as

$$\mathcal{C}^\infty(\Xi) := \{f \in W^{\infty,2}(\Xi) : \forall(\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n, \quad D_1^{\alpha_1} \dots D_n^{\alpha_n} b \in L^\infty(\Xi)\}.$$

The space $\mathcal{C}^\infty(\Xi)$ is dense in $L^2(\Xi)$ ([6], Lemma 2.1(b)) and separable ([6], Lemma 2.2). At this point we would like to emphasize, that in the stochastic setting we do not have Poincaré- or Sobolev-type estimates. Hence, the L^2-integrability of higher-order derivatives does not yield an L^∞-bound on the derivatives. Especially, the space $W^{1,2}(\Xi)$ is in general incomplete w.r.t. to the seminorm $|\cdot|_{W^{1,2}(\Xi)}$. Therefore, we introduce its completion denoted as $\mathcal{W}^{1,2}(\Xi)$. Differential operators D_k then extend uniquely as operators $W^{1,2}(\Xi) \to L^2(\Xi)$ to continuous operators $\mathcal{W}^{1,2}(\Xi) \to L^2(\Xi)$. The n-tuple of differential operators $D = (D_1, \dots, D_n)$ will be called *stochastic gradient*.

We say that elements $\widetilde{\rho} \in \Xi$ are *typical*, if the identity in the Birkhoff's ergodicity theorem (56) holds for all $g \in \mathcal{C}^\infty(\Xi)$, and a trajectory $x \mapsto T_x\widetilde{\rho}$ will be called *typical*, if $\widetilde{\rho}$ is typical. Note that separability of $\mathcal{C}^\infty(\Xi)$ implies that almost every $\rho \in \Xi$ is typical. This enables us to prove the following.

Lemma 4.2. *Let $n = 1$. Then for every $b \in L^2(\Xi)$ with $\int_\Xi b(\rho)\mathrm{d}\mathbb{P}(\rho) = 0$, there exists $g \in \mathcal{W}^{1,2}(\Xi)$ such that*

$$D_1 g = b\,.$$

Remark 4.3. *Notice that the zero mean value is necessary, since $\int D_1 g(\rho)\mathrm{d}\mathbb{P}(\rho) = 0$ for any $g \in \mathcal{W}^{1,2}(\Xi)$.*

Proof. By [12, Proposition A.9.], there exists a decomposition

$$L^2(\Xi) = F^2_{pot}(\Xi) \oplus F^2_{sol}(\Xi) \oplus \mathbb{R}\,,$$

where

$$\begin{aligned} F^2_{pot}(\Xi) &:= \mathrm{Cl}_{L^2}\{D\chi : \chi \in W^{1,2}(\Xi)\}\,, \\ F^2_{sol}(\Xi) &:= \mathrm{Cl}_{L^2}\{D \times \chi : \chi \in W^{1,2}(\Xi)\}\,. \end{aligned}$$

For $n = 1$ we have $D \times \chi = 0$ by definition, and the statement follows. □

The concept of two-scale convergence was first introduced by Nguetseng in [29] for periodic problems, while Allaire further developed the concept and methods to a versatile tool [1]. For the stochastic setting, the first definition was given in [6]. However, that concept is not well suited for our purpose and we will instead use the following (slightly altered) definitions and results given in [33].

Definition 4.3 (Weak stochastic two-scale convergence). *Let $(T_x\widetilde{\rho})_{x\in\mathbb{R}^n}$ be a typical trajectory and (v^ε) bounded sequence of functions in $L^2(\Omega)$. We say that (v^ε) stochastically weakly two-scale converges to $v^{\widetilde{\rho}} \in L^2(\Omega\times\Xi)$ w.r.t. $\widetilde{\rho}$ and we write $v^\varepsilon \overset{2}{\rightharpoonup} v^{\widetilde{\rho}}$ if*

$$\lim_{\varepsilon\downarrow 0}\int_\Omega v^\varepsilon(x)\varphi(x)b(T_{\varepsilon^{-1}x}\widetilde{\rho})\mathrm{d}x = \int_\Xi\int_\Omega v^{\widetilde{\rho}}(x,\rho)\varphi(x)b(\rho)\mathrm{d}x\mathrm{d}\mathbb{P}(\rho)$$

for all $\varphi\in C_c^\infty(\Omega)$ and $b\in\mathcal{C}^\infty(\Xi)$. Vector-valued functions are said to stochastically weakly two-scale converge, if every component stochastically weakly two-scale converges.

Remark 4.4. *The difference in this definition to the original one in [33] is the space $\mathcal{C}^\infty(\Xi)$ instead of $C^0(\Xi)$ for the test functions b. This allows us to skip the assumption of a metric on Ξ. Observe that the limit v may depend on the choice of the typical element, moreover, the sequence (v^ε) may convergence for some typical elements, while not for others. From now on we fix a typical $\widetilde{\rho}\in\Xi$ and suppress any dependence on it.*

Definition 4.4 (Strong stochastic two-scale convergence). *Let $(v^\varepsilon)\subset L^2(\Omega)$ be a weakly stochastic two-scale convergent sequence with limit $v^0\in L^2(\Omega\times\Xi)$. We say that (v^ε) converges strongly stochastic two-scale to v^0 if additionally*

$$\lim_{\varepsilon\downarrow 0}\int_\Omega v^\varepsilon(x)u^\varepsilon(x)\mathrm{d}x = \int_\Xi\int_\Omega v^0(x,\rho)u^0(x,\rho)\mathrm{d}x\mathrm{d}\mathbb{P}(\rho)$$

for every $(u^\varepsilon)\subset L^2(\Omega)$ weakly stochastically two-scale converging to $u^0\in L^2(\Omega\times\Xi)$. We denote that by $v^\varepsilon \overset{2}{\to} v^0$.

Lemma 4.5 (Extension of the test functions). *If $v^\varepsilon \overset{2}{\rightharpoonup} v$, then*

$$\lim_{\varepsilon\downarrow 0}\int_\Omega v^\varepsilon(x)\varphi(x)b(T_{\varepsilon^{-1}x_1}\widetilde{\rho})\mathrm{d}x = \int_\Xi\int_\Omega v(x,\rho)\varphi(x)b(\rho)\mathrm{d}x\mathrm{d}\mathbb{P}(\rho)$$

holds also for $b\in L^2(\Xi)$.

Lemma 4.6 (Compactness). *Let (v^ε) be a bounded sequence in $L^2(\Omega)$, then there exists a subsequence (not relabeled) and $v\in L^2(\Omega\times\Xi)$ such that $v^\varepsilon \overset{2}{\rightharpoonup} v$.*

Lemma 4.7. *Let (u^ε) be a bounded sequence in $W^{1,2}(\Omega)$. Then on a subsequence (not relabeled) $u^\varepsilon \rightharpoonup u^0$ in $W^{1,2}(\Omega)$ and there exists $u^1\in L^2(\Omega,\mathcal{W}^{1,2}(\Xi))$ such that*

$$u^\varepsilon \overset{2}{\rightharpoonup} u^0 \quad \textit{and} \quad \nabla u^\varepsilon \overset{2}{\rightharpoonup} \nabla u^0 + Du^1\,.$$

The next lemma shows that convex/quadratic functionals are compatible with this concept of two-scale convergence. A similar statement with proof can be found in [20].

Lemma 4.8 (Lower-semicontinuity and continuity of quadratic functionals). *Let (u^ε) be a bounded sequence in $L^2(\Omega,\mathbb{R}^n)$ such that $u^\varepsilon \xrightharpoonup{2} u^0 \in L^2(\Omega\times\Xi,\mathbb{R}^n)$. Let $Q:\Xi\times\mathbb{R}^n\to[0,\infty)$ be a measurable map such that for a.e. $\rho\in\Xi$, $Q(\rho,\cdot)$ is a bounded positive semidefinite quadratic form, i.e. there exists $\alpha>0$ such that*

$$|Q(\rho,v)|\le\alpha|v|^2\,,\quad \forall v\in\mathbb{R}^n\,.$$

Then

$$\lim_{\varepsilon\downarrow 0}\int_\Omega Q\big(T_{\varepsilon^{-1}x_1}\widetilde{\rho},u^\varepsilon(x)\big)\mathrm{d}x \ge \int_\Omega\int_\Xi Q\big(\rho,u^0(\rho,x)\big)\mathrm{d}\mathbb{P}(\rho)\mathrm{d}x.$$

If additionally $u^\varepsilon \xrightarrow{2} u^0$, then

$$\lim_{\varepsilon\downarrow 0}\int_\Omega Q\Big(T_{\varepsilon^{-1}x_1}\widetilde{\rho},u^k(x)\Big)\mathrm{d}x = \int_\Omega\int_\Xi Q\big(\rho,u^0(\rho,x)\big)\mathrm{d}\mathbb{P}(\rho)\mathrm{d}x.$$

4.2. Application in elasticity

In this subsection we closely follow [27], where analogous results where derived for the periodic case. Since most of the statements can be proved in the same fashion, we will skip those. In the following we work only with a one-dimensional dynamical systems T, i.e. $n=1$. We could assume additional microstructure in the cross section (see for instance [22] for the periodic case of bending plate), but for simplicity omit that.

Let (ε_h) be a sequence of positive numbers, such that $\varepsilon_h\downarrow 0$ for $h\downarrow 0$. The random energy density $W^h:\mathbb{R}^3\times\Xi\times\mathbb{R}^{3\times3}\to[0,+\infty]$ is then defined by

$$W^h(x,\rho,F)=W(T_{\varepsilon_h^{-1}x_1}\rho,F)\,, \tag{57}$$

where

(S1) for a.e. $\rho\in\Xi$, $W(\rho,\cdot)$ is continuous function on $\mathbb{R}^{3\times3}$;

(S2) for a.e. $\rho\in\Xi$, $W(\rho,\cdot)\in\mathcal{W}(\alpha,\beta,\varrho,\kappa)$;

(S3) there exists a monotone function $r:\mathbb{R}_+\to(0,+\infty)$ such that $r(\delta)\downarrow 0$ as $\delta\downarrow 0$ and

$$\forall G\in\mathbb{R}^{3\times3}\,,\ \forall h>0\ :\ \operatorname*{ess\,sup}_{x\in\Omega}|W^h(x,\rho,I+G)-Q^h(x,\rho,G)|\le r(|G|)|G|^2\,,$$

where $Q^h(x,\rho,\cdot)$ are quadratic forms defined as in (H4).

The limiting material properties depend strongly on the relation between h and ε_h, more specifically on $\gamma\in[0,+\infty]$ defined by

$$\gamma:=\lim_{h\downarrow 0}\frac{h}{\varepsilon_h}\,.$$

To study the above introduced energies we need Sobolev-type spaces not only in Ξ, but also on $\Xi\times\omega$. Hence, we define

$$W^{1,2}(\Xi\times\omega):=W^{1,2}(\omega,L^2(\Xi))\cap L^2(\omega,W^{1,2}(\Xi))\,,$$

equipped with seminorm

$$|u|^2_{W^{1,2}(\Xi\times\omega)} = \|D_1 u\|^2_{L^2(\Xi\times\omega)} + \|\partial_2 u\|^2_{L^2(\Xi\times\omega)} + \|\partial_3 u\|^2_{L^2(\Xi\times\omega)}\,.$$

Similarly as in the purely stochastic Sobolev space, by $\mathcal{W}^{1,2}(\Xi\times\omega)$ we denote the completion of $W^{1,2}(\Xi\times\omega)$ w.r.t. the seminorm $|\cdot|_{W^{1,2}(\Xi\times\omega)}$. The following statement about stochastic two-scale limit of scaled gradients can be proved as in [20].

Lemma 4.9. *Let $(u^h)\subset W^{1,2}(\Omega,\mathbb{R}^3)$ and $u^0\in L^2(\Omega,\mathbb{R}^3)$ such that $u^h\to u^0$ strongly in $L^2(\Omega,\mathbb{R}^3)$ and let $(\nabla_h u^h)$ be uniformly bounded in $L^2(\Omega,\mathbb{R}^{3\times3})$. Then u^0 depends only on x_1. Moreover,*

1. *if $\gamma\in\{0,\infty\}$, then there exists*

$$\begin{cases} u^1\in L^2((0,L),(\mathcal{W}^{1,2}(\Xi))^3) \text{ and } u^2\in L^2((0,L)\times\Xi, W^{1,2}(\omega,\mathbb{R}^3))\,, & \gamma=0\,,\\ u^1\in L^2(\Omega,(\mathcal{W}^{1,2}(\Xi))^3) \text{ and } u^2\in L^2(I, W^{1,2}(\omega,\mathbb{R}^3))\,, & \gamma=\infty\,,\end{cases}$$

and

$$\nabla_h u^h \xrightharpoonup{2} (\partial_1 u^0 + D_1 u^1 \,|\, \nabla_{x'} u^2)\,.$$

2. *If $\gamma\in(0,\infty)$, then there is a subsequence (not relabeled) and a function $u^1\in L^2((0,L),\mathcal{W}^{1,2}(\Xi\times\omega,\mathbb{R}^3))$ such that*

$$\nabla_h u^h \xrightharpoonup{2} (\partial_1 u^0 + D_1 u^1 \,|\, \frac{1}{\gamma}\nabla_{x'} u^1)\,.$$

4.3. Cell formula

Definition 4.5. *For a.e. $\rho\in\Xi$ let $Q(\rho,\cdot)$ be a quadratic form associated with the energy density $W(\rho,\cdot)$. For every $\varrho\in\mathbb{R}$ and $\Psi\in\mathbb{R}^{3\times3}_{\mathrm{skw}}$, define the mapping $Q^0_\gamma:\mathbb{R}\times\mathbb{R}^3\to\mathbb{R}$ by*

$$Q^0_\gamma(\varrho,\operatorname{axl}\Psi) := \begin{cases} \inf \int_\Xi\int_\omega Q\big(\rho,\iota(\varrho e_1+\Psi\mathfrak{p}_{x'}+(D_1\Psi^1)\mathfrak{p}_{x'})+(D_1\vartheta^1\,|\,\nabla_{x'}\vartheta^2)\big)\mathrm{d}x'\mathrm{d}\mathbb{P}(\rho)\,, & \gamma=0\,;\\ \inf \int_\Xi\int_\omega Q\big(\rho,\iota(\varrho e_1+\Psi\mathfrak{p}_{x'})+\big(D_1\vartheta^1\,|\,\tfrac{1}{\gamma}\nabla_{x'}\vartheta^1\big)\big)\mathrm{d}x'\mathrm{d}\mathbb{P}(\rho)\,, & 0<\gamma<\infty\,;\\ \inf \int_\Xi\int_\omega Q\big(\rho,\iota(\varrho e_1+\Psi\mathfrak{p}_{x'})+(D_1\vartheta^1\,|\,\nabla_{x'}\vartheta^2)\big)\mathrm{d}x'\mathrm{d}\mathbb{P}(\rho)\,, & \gamma=\infty\,,\end{cases}$$

where the infimum is taken over all $\Psi^1,\vartheta^1,\vartheta^2$ satisfying: $\Psi^1\in\mathcal{W}^{1,2}(\Xi,\mathbb{R}^{3\times3}_{\mathrm{skw}})$,

$$\vartheta^1\in\begin{cases}\mathcal{W}^{1,2}(\Xi)^3\,, & \gamma=0\,,\\ \mathcal{W}^{1,2}(\Xi\times\omega)^3\,, & 0<\gamma<\infty\,,\\ L^2(\omega,\mathcal{W}^{1,2}(\Xi)^3)\,, & \gamma=\infty\,,\end{cases} \quad \text{and} \quad \vartheta^2\in\begin{cases} L^2(\Xi,W^{1,2}(\omega,\mathbb{R}^3))\,, & \gamma=0\,,\\ W^{1,2}(\omega,\mathbb{R}^3)\,, & \gamma=\infty\,.\end{cases}$$

Proposition 4.10. *Let (W^h) be a family of energy densities describing a random material for rods defined by (57). Then the limit energy density Q^0, defined in (24), is given by Q^0_γ from Definition 4.5.*

Proof. We only prove the result for $0<\gamma<\infty$. The other two cases are very similar. Using the previous general homogenization result it suffices to prove that for $m=m(\varrho,\Psi)=\varrho e_1+\Psi\mathfrak{p}_{x'}$ it holds

$$\lim_{r\downarrow0}\Big(\frac{1}{2r}\mathcal{K}_{(h)}(m,x_1^0+(-r,r))\Big) = Q^0_\gamma(\varrho,\operatorname{axl}\Psi)\,,$$

for all $\varrho \in \mathbb{R}$ and $\Psi \in \mathbb{R}^{3\times 3}_{\rm skw}$, for every Lebesgue point x_1^0, where $\mathcal{K}_{(h)}$ is given by (22). By Lemma 2.5, for given m, there exist sequences of functions $(\Psi^h) \subset H^1((0,L),\mathbb{R}^{3\times 3}_{\rm skw})$ and $(\vartheta^h) \subset H^1(\Omega,\mathbb{R}^3)$, with properties stated there, such that

$$\lim_{r\downarrow 0}\Big(\frac{1}{2r}\mathcal{K}_{(h)}(m,x_1^0+(-r,r))\Big) = \\ \lim_{r\downarrow 0}\frac{1}{2r}\lim_{h\downarrow 0}\int_{(x_1^0+(-r,r))\times\omega} Q\Big(T_{\varepsilon^{-1}x_1}\widetilde{\rho},\iota(m)+\operatorname{sym}\iota((\Psi^h)'\mathfrak{p}_{x'})+\operatorname{sym}\nabla_h\vartheta^h\Big)\mathrm{d}x\,.$$

Using the lower-semicontinuity of quadratic functionals with respect to the stochastic two-scale convergence we obtain

$$\begin{aligned}\lim_{r\downarrow 0}&\Big(\frac{1}{2r}\mathcal{K}_{(h)}(m,x_1^0+(-r,r))\Big)\\ &=\lim_{r\downarrow 0}\frac{1}{2r}\lim_{h\downarrow 0}\int_{(x_1^0+(-r,r))\times\omega} Q\Big(T_{\varepsilon^{-1}x_1}\widetilde{\rho},\iota(m)+\operatorname{sym}((\Psi^h)'\mathfrak{p}_{x'})+\operatorname{sym}\nabla_h\vartheta^h\Big)\mathrm{d}x\\ &\geq \lim_{r\downarrow 0}\frac{1}{2r}\inf_U\int_{(x_1^0+(-r,r))}\int_{\Xi\times\omega} Q\Big(\rho,\iota(m)+U\Big)\mathrm{d}\mathbb{P}(\rho)\mathrm{d}x\,,\end{aligned}$$

where the infimum is taken over all possible two-scale limits of

$$\operatorname{sym}\iota((\Psi^h)'\mathfrak{p}_{x'})+\operatorname{sym}\nabla_h\vartheta^h\,,$$

i.e.

$$\Big\{\operatorname{sym}\iota(D_1\Psi^1\mathfrak{p}_{x'})+\operatorname{sym}\frac{1}{\gamma}\nabla_{x'}\vartheta^1 \;:\; \Psi^1\in\mathcal{W}^{1,2}(\Xi,\mathbb{R}^{3\times 3}_{skw})\,,\ \vartheta^1\in\mathcal{W}^{1,2}(\Xi\times\omega)^3\Big\}\,.$$

Notice that the first term can be absorbed into the second one. To show this we define $\widetilde{\vartheta}^1$ by

$$\widetilde{\vartheta}^1(\rho,x') := \begin{pmatrix}\Psi_{12}(\rho)x_2+\Psi_{13}(\rho)x_3\\ -\frac{1}{\gamma}\widehat{\Psi}_{12}(\rho)+\Psi_{23}(\rho)x_3\\ -\frac{1}{\gamma}\widehat{\Psi}_{13}(\rho)-\Psi_{23}(\rho)x_2\end{pmatrix}\,,$$

where $\widehat{\cdot}$ denotes the primitive of the function. A short calculation reveals that

$$\operatorname{sym}(\frac{1}{\gamma}\nabla_{x'}\widetilde{\vartheta}^1)=\operatorname{sym}((D_1\Psi)\mathfrak{p}_{x'})\,.$$

Therefore, the set of weak stochastic two-scale limits is given by

$$\Big\{\frac{1}{\gamma}\nabla_{x'}\vartheta^1 \;:\; \vartheta^1\in(\mathcal{W}^{1,2}(\Xi\times\omega))^3\Big\}\,.$$

Hence, we deduce

$$\lim_{r\downarrow 0}\Big(\frac{1}{2r}\mathcal{K}_{(h)}(m,x_1^0+(-r,r))\Big)\geq Q^0_\gamma(\varrho,\operatorname{axl}\Psi)\,.$$

For the reverse inequality we fix ϱ,Ψ, and let $\vartheta^1\in(W^{1,2}(\Xi\times\omega))^3$ be such that

$$\int_\Xi\int_\omega Q\Big(\rho,\iota(\varrho e_1+\Psi\mathfrak{p}_{x'})+\big(D_1\vartheta^1\,|\,\frac{1}{\gamma}\nabla_{x'}\vartheta^1\big)\Big)\mathrm{d}x'\mathrm{d}\mathbb{P}(\rho)\leq\varepsilon+Q^0_\gamma(\varrho,\operatorname{axl}\Psi)\,.$$

Defining

$$\vartheta^h(x_1,x') = \frac{h}{\gamma}\vartheta^1(T_{\varepsilon^{-1}x_1}\widetilde{\rho},x')\,,$$

we observe

$$\mathrm{sym}\big(\nabla_h\vartheta^h\big) \xrightarrow{2} \mathrm{sym}\Big(D_1\vartheta^1\,|\,\frac{1}{\gamma}\nabla_{x'}\vartheta^1\Big)\,.$$

By continuity of quadratic functions w.r.t. stochastic two-scale convergence we have

$$\begin{aligned}
&\varepsilon + Q^0_\gamma(\varrho,\mathrm{axl}\,\Psi)\\
&\quad\geq \int_\Xi\int_\omega Q\Big(\rho,\iota(\varrho e_1+\Psi\mathfrak{p}_{x'}) + \mathrm{sym}\big(D_1\vartheta^1\,|\,\frac{1}{\gamma}\nabla_{x'}\vartheta^1\big)\Big)\mathrm{d}x'\mathrm{d}\mathbb{P}(\rho)\\
&\quad= \lim_{r\downarrow 0}\frac{1}{2r}\int_{x_1+(-r,r)}\lim_{h\downarrow 0}\int_\omega Q\Big(T_{\varepsilon^{-1}x_1}\widetilde{\rho},\iota(\rho e_1+\Psi\mathfrak{p}_{x'}) + \mathrm{sym}\big(\nabla_h\vartheta^h\big)\Big)\mathrm{d}x'\mathrm{d}\mathbb{P}(\rho)\,,
\end{aligned}$$

which finishes the proof. □

Appendix

Lemma A.1. *Let $p>1$, $\Omega\subset\mathbb{R}^d$ open, bounded set and $(u_k)\subset W^{1,p}(\Omega,\mathbb{R}^m)$ a bounded sequence such that $(|\nabla u_k|^p)$ is equi-integrable. Let $(s_k)_k$ be an increasing sequence of positive reals such that $s_k\to+\infty$ for $k\to+\infty$. Then there exists a subsequence still denoted by (u_k) and a sequence $(z_k)\subset W^{1,\infty}(\Omega,\mathbb{R}^m)$ satisfying: $|z_k\neq u_k|\to 0$ as $k\to+\infty$, $(|\nabla z_k|^p)$ is equi-integrable and $\|z_k\|_{W^{1,\infty}}\leq Cs_k$ for some $C>0$ depending only on dimension d.*

Proof. The proof is implicitly contained in the proof of Lemma 1.2 (decomposition lemma) from [16], but we include it here for reasons of completeness. As in [16], the proof is divided into two steps. In the first we assume that Ω is an extension domain, while in the second we remove this restriction generalizing the statement for an arbitrary open set.

Step 1. Let $\Omega\subset\mathbb{R}^d$ be an extension domain, i.e. an open, bounded set for which there exists an extension operator $T_\Omega : W^{1,p}(\Omega,\mathbb{R}^m)\to W^{1,p}(\mathbb{R}^d,\mathbb{R}^m)$ satisfying:

$$T_\Omega u = u \quad \text{on } \Omega\,, \qquad \|T_\Omega u\|_{W^{1,p}(\mathbb{R}^d)} \leq C\|u\|_{W^{1,p}(\Omega)}\,.$$

In the following we identify the sequence $(u_k)\subset W^{1,p}(\Omega,\mathbb{R}^m)$ with its extension sequence $(T_\Omega u_k)\subset W^{1,p}(\mathbb{R}^d,\mathbb{R}^m)$. Let us introduce the Hardy-Littlewood maximal function

$$M(u)(x) := \sup_{r>0}\frac{1}{|B(x,r)|}\int_{B(x,r)}|u(y)|\mathrm{d}y\,,$$

defined for any Borel measurable function $u:\mathbb{R}^d\to\mathbb{R}^m$. It is known that for $p>1$ and $u\in W^{1,p}(\mathbb{R}^d,\mathbb{R}^m)$,

$$\|M(u)\|_{L^p} + \|M(\nabla u)\|_{L^p} \leq C\|u\|_{W^{1,p}}\,.$$

According to [16, Lemma 4.1] (cf. [13]), for every $k\in\mathbb{N}$, there exists $z_k\in W^{1,\infty}(\mathbb{R}^d,\mathbb{R}^m)$ such that $u_k = z_k$ on the set $\mathcal{S}_k := \{M(\nabla u_k)(x) < s_k\}$ and $\|z_k\|_{W^{1,\infty}}\leq Cs_k$, where $C>0$ depends only on d. Using the argument as in the proof of [14, Proposition A2.], we obtain an estimate on the Lebesgue measure of the complement of set $\mathcal{S}_k$,

$$|\mathcal{S}_k^c| \leq \frac{C}{s_k^p}\int_{\{|u_k|+|\nabla u_k|>s_k/2\}}(|u_k|+|\nabla u_k|)^p\mathrm{d}x\,, \quad \text{for all } k\in\mathbb{N}\,. \tag{58}$$

The strong convergence of (u_k) and the equi-integrability property of $(|\nabla u_k|^p)$ imply that $|\mathcal{S}_k^c| = |u_k \neq z_k| \to 0$ as $k \to \infty$. Let $A \subset \mathbb{R}^d$ be a bounded open subset, then due to the fact that $\{u_k = z_k\} = \{u_k = z_k\,, \nabla u_k = \nabla z_k\}$, up to a set of the Lebesgue measure zero, we have

$$\int_A |\nabla z_k|^p \mathrm{d}x = \int_{A\cap\mathcal{S}_k} |\nabla u_k|^p \mathrm{d}x + \int_{A\cap\mathcal{S}_k^c} |\nabla z_k|^p \mathrm{d}x\,, \quad \text{for all } k \in \mathbb{N}\,.$$

Since $(|\nabla u_k|^p)$ is equi-integrable, the first term on the right-hand side can be made arbitrary small for $|A|$ small enough. For the second term, using (58), we estimate

$$\int_{\mathcal{S}_k^c} |\nabla z_k|^p \mathrm{d}x \le \|\nabla z_k\|_{L^\infty}^p |\mathcal{S}_k^c| \le C \int_{\{|u_k|+|\nabla u_k|>s_k/2\}} (|u_k| + |\nabla u_k|)^p \mathrm{d}x\,, \quad \text{for all } k \in \mathbb{N}\,,$$

and conclude, as above, that $\lim_{k\to\infty} \int_{\mathcal{S}_k^c} |\nabla z_k|^p \mathrm{d}x = 0$. Hence, we proved that for every $\varepsilon > 0$ there exists $\delta > 0$ and $k_0 \in \mathbb{N}$, such that for all open subsets $A \subset \mathbb{R}^d$ with $|A| \le \delta$ and for all $k \ge k_0$ it holds

$$\int_A |\nabla z_k|^p \mathrm{d}x \le \varepsilon\,,$$

which by definition means the equi-integrability of the sequence $(|\nabla z_k|^p)$.

Step 2. Let Ω be an arbitrary open, bounded set. For a given bounded sequence $(u_k) \subset W^{1,p}(\Omega, \mathbb{R}^m)$, there exists a subsequence such that

$$u_k \rightharpoonup u \quad \text{in } W^{1,p}(\Omega, \mathbb{R}^m)\,, \quad u_k \to u \quad \text{in } L^p_{loc}(\Omega, \mathbb{R}^m)\,.$$

Let (Ω_l) be an increasing sequence of compactly contained subdomains of Ω satisfying $|\Omega\backslash\Omega_l| \to 0$ as $l \to \infty$, and let $(\zeta_l) \subset C_c^\infty(\Omega, [0,1])$ be a sequence of cut-off functions such that $\zeta_l(x) = 1$ for $x \in \Omega_l$. Define $\tilde{u}_k := u_k - u$, and observe that

$$\limsup_{l\to\infty} \limsup_{k\to\infty} \|\zeta_l \tilde{u}_k\|_{L^p} = 0$$

and

$$\begin{aligned}\limsup_{l\to\infty} \limsup_{k\to\infty} \|\nabla(\zeta_l \tilde{u}_k)\|_{L^p} &= \limsup_{l\to\infty} \limsup_{k\to\infty} \|\nabla\zeta_l \otimes \tilde{u}_k + \zeta_l \nabla\tilde{u}_k\|_{L^p} \\ &\le \limsup_{k\to\infty} \|\nabla\tilde{u}_k\|_{L^p} < \infty\,.\end{aligned}$$

Then, a standard diagonalization procedure applies (cf. [2, Lemma 1.15]) and provides a bounded sequence $(\zeta_{l(k)}\tilde{u}_k) \subset W_0^{1,p}(\Omega, \mathbb{R}^m)$, which can be extended by zero to $\mathbb{R}^d$. Since, $(|\nabla(\zeta_{l(k)}\tilde{u}_k)|^p)$ is equi-integrable, applying the arguments of Step 1, there exists a sequence $(\tilde{z}_k) \subset W^{1,p}(\Omega, \mathbb{R}^m)$ satisfying: $|\tilde{z}_k \neq \zeta_{l(k)}\tilde{u}_k| \to 0$ as $k \to +\infty$, $(|\nabla\tilde{z}_k|^p)$ is equi-integrable and $\|\tilde{z}_k\|_{W^{1,\infty}} \le Cs_k$ for some $C > 0$. Since, $|\tilde{z}_k + u \neq u_k| \le |\tilde{z}_k \neq \zeta_{l(k)}\tilde{u}_k| + |\Omega\backslash\Omega_{l(k)}| \to 0$, $(|\nabla(\tilde{z}_k + u)|^p)$ is equi-integrable, and $\|\tilde{z}_k + u\|_{W^{1,\infty}} \le Cs_k$ for some $C > 0$, we identify $z_k = \tilde{z}_k + u$ as the sought sequence. □

Remark A.2. *If we assume in the previous lemma that Ω is a Lipschitz domain, as it is the case in our model of the rod, where $\Omega = (0, L) \times \omega$ and ω is Lipschitz, then Ω is also an extension domain and according to the arguments in Step 1, we can replace the whole sequence (u_k) by its Lipschitz counterpart.*

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